Macroeconometric modelling

6 Forecasting and dynamic analysis

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Introduction

- Dynamic multipliers provide a simple way to describe the properties of a dynamic model. See Lütkepohl (2005) for a general treatment.
- Integratedness of the data poses no difficulties as long as the model can be cast in I(0) form.
- As an example, we will work out the dynamic properties of the wage-price model in Bårdsen, Eitrheim, Jansen, and Nymoen (2005):

$$\begin{aligned}
\Delta w_t = &\underset{(0.11)}{0.81}\ \Delta p_t - \underset{(0.12)}{0.51}\ \Delta h_t + \underset{(0.017)}{0.082}\ \Delta pr_t \\
&- \underset{(0.021)}{0.16}\ eqcw_t - \underset{(0.0027)}{0.02}\ wdum_t + \underset{(0.0037)}{0.023}\ pdum_t \\
&+ \underset{(0.0021)}{0.024}\ CS2_t - \underset{(0.017)}{0.12}
\end{aligned}$$

$$\begin{aligned}
\Delta p_t = &\underset{(0.027)}{0.14}\ \Delta w_t + \underset{(0.021)}{0.1}\ \Delta w_{t-1} + \underset{(0.048)}{0.16}\ \Delta p_{t-2} \\
&- \underset{(0.0059)}{0.015}\ \Delta pr_t + \underset{(0.0081)}{0.026}\ \Delta pm_t + \underset{(0.0075)}{0.042}\ \Delta pe_t \\
&- \underset{(0.0061)}{0.055}\ eqcp_t + \underset{(0.012)}{0.028}\ \Delta y_{t-1} + \underset{(0.012)}{0.046}\ \Delta y_{t-2} \\
&- \underset{(0.0011)}{0.013}\ pdum_t + \underset{(0.00092)}{0.0064}
\end{aligned}$$

This involves evalutating the stability of the model, and the long-run and dynamic multipliers. Disregarding taxes and short-run effects, the systematic part of the model is on matrix form:

$$\begin{bmatrix} 1 & -0.81 \\ -0.14 & 1 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta p \end{bmatrix}_t = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta p \end{bmatrix}_{t-1}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0.16 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta p \end{bmatrix}_{t-2} + \begin{bmatrix} 0.082 & 0 & 0 \\ -0.015 & 0 & 0.026 \end{bmatrix} \begin{bmatrix} \Delta pr \\ \Delta u \\ \Delta pm \end{bmatrix}_t$$

$$+ \begin{bmatrix} -0.16 & 0 \\ 0 & -0.055 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 0.1L & 0 \\ -0.7L & 1L^2 & 0.7 & 0 & -0.3 \end{bmatrix} \begin{bmatrix} w \\ p \\ pr \\ u \\ pm \end{bmatrix}_{t-1}$$

Dynamic properties from cointegration

The long-run elasticities of the model is, from the cointegration analysis:

$$w = p + pr - 0.1u$$
$$p = 0.7(w - pr) + 0.3pm,$$

so the long-run multipliers of the system should be easily obtained by solving for wages and prices. For wages:

$$\begin{aligned} w &= 0.7(w - pr) + 0.3p + pr - 0.1u \\ w\,(1 - 0.7) &= -0.7pr + 0.3pm + pr - 0.1u \\ w &= \frac{0.3}{0.3}pr - \frac{0.1}{0.3}u + pm \\ w &= pr - 0.33u + pm. \end{aligned}$$

Then for prices:

$$\begin{aligned} p &= 0.7(w - pr) + 0.3pm \\ &= 0.7\,(-0.33u + pm) + 0.3pm \\ p &= -0.23u + pm \end{aligned}$$

So the reduced form long-run multipliers of wages and prices with respect to the exogenous variables are

$$w = pr - 0.33u + pm$$
$$p = -0.23u + pm.$$

Note that the long-run multipliers of the real wage is given from the wage curve alone

$$w - p = pr - 0.1u$$

Imposing long-run properties of exogenous variables

- $\Delta pr = g_{pr}$
- $\Delta u = 0$
- $\Delta pm = g_{pm}$

gives the long-run multipliers for inflation

$$\pi = g_p = \Delta p = g_{pm}.$$

Finally, the steady-state growth path of the nominal system is

$$g_w = g_{pr} + g_{pm}$$
$$g_p = g_{pm}$$

Dynamic properties from difference equations

Now, let us try to see if this holds for the dynamic system. Intuitively, the same steady-state — and therefore multipliers — should be obtained if no invalid restrictions are imposed.

For the dynamic analysis of the system below, it will be more convenient to work with the model in lag-polynomial form $\mathbf{A}(L)\,\mathbf{y}_t = \mathbf{B}(L)\,\mathbf{x}_t$. This is easily achieved with the steps:

$$\begin{bmatrix} 1 & -0.81 \\ -0.14 - 0.1\,L & 1 - 0.16L^2 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta p \end{bmatrix}_t = \begin{bmatrix} 0.082 & 0 & 0 \\ -0.015 & 0 & 0.026 \end{bmatrix} \begin{bmatrix} \Delta pr \\ \Delta u \\ \Delta pm \end{bmatrix}_t$$

$$+ \begin{bmatrix} -0.16 & 0 \\ 0 & -0.055 \end{bmatrix} \begin{bmatrix} L & -L & -L & 0.1L^2 & 0 \\ -0.7L^2 & L^3 & 0.7L & 0 & -0.3L \end{bmatrix} \begin{bmatrix} w \\ p \\ pr \\ u \\ pm \end{bmatrix}_t$$

Dynamic properties from difference equations II

or:

$$\begin{bmatrix} 1-1L & -0.81+0.81L \\ -0.14-0.1\,L-\left(-0.14-0.1\,L\right)L & 1-0.16L^2-\left(1-0.16L^2\right)L \end{bmatrix} \begin{bmatrix} w \\ p \end{bmatrix}_t =$$

$$+\begin{bmatrix} 0.082-0.082L & 0 & 0 \\ -0.015+0.015L & 0 & 0.026-0.026L \end{bmatrix} \begin{bmatrix} pr \\ u \\ pm \end{bmatrix}_t$$

$$+\begin{bmatrix} -0.16L & 0.16L & 0.16L & -0.016\,L^2 & 0 \\ 0.038\,5L^2 & -0.055\,L^3 & -0.038\,5L & 0 & 0.016\,5L \end{bmatrix} \begin{bmatrix} w \\ p \\ pr \\ u \\ pm \end{bmatrix}_t$$

Dynamic properties from difference equations III

and collecting terms:

$$\underbrace{\begin{bmatrix} 1-0.84L & -0.81+0.65L \\ -0.14+0.04L+0.0615L^2 & 1-0.16L^2-1L+0.215\,L^3 \end{bmatrix}}_{\mathbf{A}(L)} \underbrace{\begin{bmatrix} w \\ p \end{bmatrix}_t}_{\mathbf{y}_t} =$$

$$+\underbrace{\begin{bmatrix} 0.082+0.078\,L & -0.016\,L^2 & 0 \\ -0.015\;-0.0235L & 0 & 0.026\;-0.0095L \end{bmatrix}}_{\mathbf{B}(L)} \underbrace{\begin{bmatrix} pr \\ u \\ pm \end{bmatrix}_t}_{\mathbf{x}_t}$$

Dynamic properties from difference equations IV

Checking stability

For the system to be stable, the autoregressive part need to have all roots outside the unit circle.

The autoregressive polynomial is

$$\mathbf{A}(L) = \begin{bmatrix} 1 - 0.84L & -0.81 + 0.65L \\ -0.14 + 0.04L + 0.0615L^2 & 1 - 0.16L^2 - 1L + 0.215\,L^3 \end{bmatrix},$$

with determinant:

$$|\mathbf{A}(L)| = 0.886\,6 - 1.716\,6L + 0.703\,815\,L^2 + 0.309\,425\,L^3 - 0.180\,6L^4.$$

The model is stable if all the roots of

$$0.886\,6 - 1.\,716\,6z + 0.703\,815\,z^2 + 0.309\,425\,z^3 - 0.180\,6z^4 = 0$$

are outside the unit circle.

Dynamic properties from difference equations V

Checking stability

Here the polynomial can be factored (approximately) as

$$-0.1806\,(z+2.26942781)\,(z-1.03041478)\,(z-1.19380201)\,(z-1.75852774)$$

so the roots are $\begin{pmatrix} -2.26942781 \\ 1.03041478 \\ 1.19380201 \\ 1.75852774 \end{pmatrix}$.

So all roots of $|\mathbf{A}(z)| = 0$ are outside the unit circle. Also, in this case, the roots are real, so the adjustment from a shock back towards steady state will be monotonic and non-cyclical.

Dynamic properties from difference equations VI

Deriving the long-run multipliers—the hard way

Next the long-run multipliers are $\mathbf{A}^{-1}(1)\,\mathbf{B}(1)$. Here $\mathbf{A}(1)$ is given as:

$$\mathbf{A}(1) = \begin{bmatrix} 1 - 0.84 & -0.81 + 0.65 \\ -0.14 + 0.04 + 0.0615 & 1 - 0.16 - 1 + 0.215 \end{bmatrix}$$
$$= \begin{bmatrix} 0.16 & -0.16 \\ -0.0385 & 0.055 \end{bmatrix},$$

while

$$\mathbf{B}(1) = \begin{bmatrix} 0.082 + 0.078 & -0.016 & 0 \\ -0.015 - 0.0235 & 0 & 0.026 - 0.0095 \end{bmatrix}$$
$$= \begin{bmatrix} 0.16 & -0.016 & 0 \\ -0.0385 & 0 & 0.0165 \end{bmatrix}$$

Dynamic properties from difference equations VII

Deriving the long-run multipliers—the hard way

giving the long-run multipliers

$$\mathbf{A}^{-1}(1)\,\mathbf{B}(1) = \begin{bmatrix} 0.16 & -0.16 \\ -0.0385 & 0.055 \end{bmatrix}^{-1} \begin{bmatrix} 0.16 & -0.016 & 0 \\ -0.0385 & 0 & 0.0165 \end{bmatrix}$$
$$= \begin{bmatrix} 1.0 & -0.33 & 1.0 \\ 0 & -0.23 & 1.0 \end{bmatrix}$$

or

$$\begin{bmatrix} w \\ p \end{bmatrix} = \begin{bmatrix} 1.0 & -0.33 & 1.0 \\ 0 & -0.23 & 1.0 \end{bmatrix} \begin{bmatrix} pr \\ u \\ pm \end{bmatrix}$$

which corresponds to the long-run multipliers derived directly from the cointegration analysis.

So the cointegration relationships is therefore the steady-state of the dynamic system; it ties down the long-run solution of the dynamic system, and the comparative static properties—the long-run multipliers. In fact, this is nothing else than Samuelson's correspondence principle in disguise.

Deriving the long-run multipliers—the easy way

To show that cointegration is nothing but steady-state with growing variables is just finding the long-run multipliers for systems. The reduced form of the model is:

$$\begin{bmatrix} \Delta w \\ \Delta p \end{bmatrix}_t = \begin{bmatrix} 0.09 & 0 \\ 0.113 & 0 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta p \end{bmatrix}_{t-1}$$

$$\begin{bmatrix} 0 & 0.146 \\ 0 & 0.18 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta p \end{bmatrix}_{t-2} + \begin{bmatrix} 0.079 & 0 & 0.024 \\ -0.004 & 0 & 0.029 \end{bmatrix} \begin{bmatrix} \Delta pr \\ \Delta u \\ \Delta pm \end{bmatrix}_t$$

$$+\Pi(L) \begin{bmatrix} w \\ p \\ pr \\ u \\ pm \end{bmatrix}_{t-1}$$

Deriving the long-run multipliers—the easy way II

with the cointegration part alone:

$$\begin{bmatrix} -0.18+0.035\,L & 0.18-0.05L^2 & 0.145 & -0.018\,L & 0.015 \\ -0.025\ +0.042\,L & 0.025\ -0.06L^2 & -0.017 & -0.002\,5L & 0.018 \end{bmatrix} \begin{bmatrix} w \\ p \\ pr \\ u \\ pm \end{bmatrix}_{t-1},$$

or when evaluated at the same date, so in steady-state:

$$\begin{bmatrix} -0.145 & 0.13 & 0.145 & -0.018 & 0.015 \\ 0.017 & -0.035 & -0.017 & -0.002\,5 & 0.018 \end{bmatrix} \begin{bmatrix} w \\ p \\ pr \\ u \\ pm \end{bmatrix}_{t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Deriving the long-run multipliers—the easy way III

The long-run multipliers are therefore simply:

$$\begin{bmatrix} w \\ p \end{bmatrix} = -\begin{bmatrix} -0.145 & 0.13 \\ 0.017 & -0.035 \end{bmatrix}^{-1} \begin{bmatrix} 0.145 & -0.018 & 0.015 \\ -0.017 & -0.0025 & 0.018 \end{bmatrix} \begin{bmatrix} pr \\ u \\ pm \end{bmatrix}$$

$$\begin{bmatrix} w \\ p \end{bmatrix} = \begin{bmatrix} -1 & 0.33 & -1 \\ -0 & 0.23 & -1 \end{bmatrix} \begin{bmatrix} pr \\ u \\ pm \end{bmatrix},$$

as before.

Dynamic multipliers I

The dynamic multipliers of the model are given as

$$\begin{aligned}\mathbf{A}^{-1}(L)\,\mathbf{B}(L) &= \mathbf{D}(L)\\ &= \begin{bmatrix} \delta_{11}(L) & \delta_{12}(L) & \delta_{13}(L) \\ \delta_{21}(L) & \delta_{22}(L) & \delta_{23}(L) \end{bmatrix},\end{aligned}$$

while the interim multipliers are the sums of the dynamic multipliers.
The simplest solution is to match coefficients of $\mathbf{B}(L) = \mathbf{A}(L)\,\mathbf{D}(L)$ for powers of L and solve for $\delta(L)$.
Let's assume we are only interested in the first three dynamic and interim multipliers of productivity on wages:

$$\delta_{11}(L) = \delta_{11,0} + \delta_{11,1}L + \delta_{11,2}L^2. \tag{1}$$

Dynamic multipliers II

The inverse autoregressive matrix polynomials are of course the product of the inverse of the determinant and the adjoint

$$\mathbf{A}^{-1}(L) = \begin{bmatrix} 1 - 0.84L & -0.81 + 0.65L \\ -0.14 + 0.04L + 0.0615L^2 & 1 - 0.16L^2 - 1L + 0.215\,L^3 \end{bmatrix}^{-1}$$

$$= \frac{1}{0.89 - 1.72L + 0.7L^2 + 0.31L^3 - 0.18L^4}$$

$$\times \begin{bmatrix} 1 - 0.16L^2 - L + 0.215\,L^3 & 0.81 - 0.65L \\ 0.14 - 0.04L - 0.0615L^2 & 1 - 0.84L \end{bmatrix}.$$

The matrix of distributed lag polynomials was

$$\mathbf{B}(L) = \begin{bmatrix} 0.082 + 0.078\,L & -0.016\,L^2 & 0 \\ -0.015 - 0.0235L & 0 & 0.026 - 0.0095L \end{bmatrix}.$$

Dynamic multipliers III

Therefore

$$\mathbf{D}(L) = \begin{bmatrix} \delta_{11}(L) & \delta_{12}(L) & \delta_{13}(L) \\ \delta_{21}(L) & \delta_{22}(L) & \delta_{23}(L) \end{bmatrix} =$$

$$\frac{1}{0.89 - 1.72L + 0.7L^2 + 0.31\,L^3 - 0.18L^4} \begin{bmatrix} 1 - L - 0.16L^2 + 0.215\,L^3 & 0.81 - 0.65L \\ 0.14 - 0.04L - 0.061\,5L^2 & 1 - 0.84L \end{bmatrix}$$

$$\times \begin{bmatrix} 0.082 + 0.078\,L & -0.016\,L^2 & 0 \\ -0.015\ - 0.0235L & 0 & 0.026\ - 0.0095L \end{bmatrix}$$

$$= \frac{1}{0.89 - 1.72L + 0.7L^2 + 0.31\,L^3 - 0.18L^4} \times$$

$$\begin{bmatrix} 0.07 - 0.01L - 0.08L^2 + 0.01L^3 + 0.02L^4 & -0.02\,L^2 + 0.02\,L^3 + 0.003L^4 - 0.003L^5 & 0.02 - 0.02\,L + 0.006\,L^2 \\ -0.004 - 0.003L + 0.01\,L^2 - 0.005\,L^3 & -0.002L^2 + 0.0006L^3 + 0.001\,L^4 & 0.03 - 0.03L + 0.008L^2 \end{bmatrix}$$

Dynamic multipliers IV

So to find the dynamic multipliers of wages with respect to productivity $\delta_{11,i}$, for period $i = 0, 1, 2$, we have to solve

$$
\begin{aligned}
&0.07 - 0.013L - 0.076L^2 + 0.005L^3 + 0.02L^4 \\
&= \left(0.89 - 1.72L + 0.7\,L^2 + 0.31\,L^3 - 0.18L^4\right)\left(\delta_{11,0} + \delta_{11,1}L + \delta_{11,2}L^2\right) \\
&= 0.89\delta_{11,0} + (0.89\delta_{11,1} - 1.72\delta_{11,0})\,L + (0.89\delta_{11,2} - 1.72\delta_{11,1} + 0.70\delta_{11,0})\,L^2 \\
&\quad + (-1.72\delta_{11,2} + 0.70\,\delta_{11,1} + 0.31\,\delta_{11,0})\,L^3 \\
&\quad + (0.70\,\delta_{11,2} + 0.31\,\delta_{11,1} - 0.18\delta_{11,0})\,L^4 + (0.31\delta_{11,2} - 0.18\delta_{11,1})\,L^5 - 0.18\delta_{11,2}L^6
\end{aligned}
$$

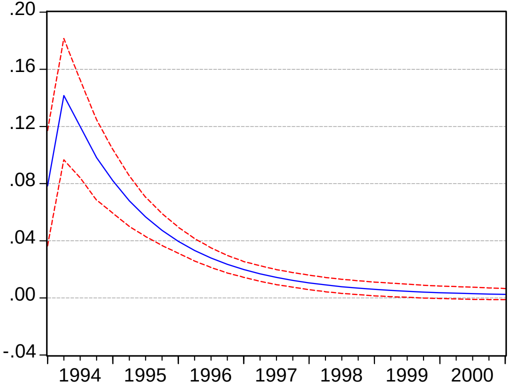
for the δ's by evaluating the polynomials for powers of L:

$$
\begin{aligned}
L = 0: &\quad \delta_{11,0} = \tfrac{0.07}{0.89} = 0.079 \\
L = 1: &\quad \delta_{11,1} = \tfrac{1.72\delta_{11,0} - 0.013}{0.89} = 0.138 \\
L = 2: &\quad \delta_{11,2} = \tfrac{1.72\delta_{11,1} - 0.70\delta_{11,0} - 0.076}{0.89} = 0.119
\end{aligned}
$$

Dynamic multipliers V

Graphically, the result is

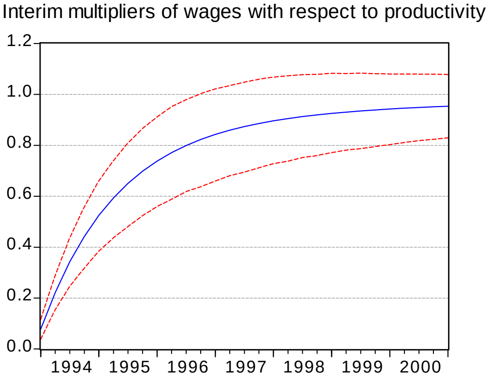
Dynamic multipliers for wage with respect to productivity



Dynamic multipliers VI

The interim multipliers are the partial sums of the dynamic multipliers:

$$\delta^*_{11,0} = \delta_{11,0} = 0.079$$
$$\delta^*_{11,1} = \delta^*_{11,0} + \delta_{11,1} = 0.079 + 0.138 = 0.217$$
$$\delta^*_{11,2} = \delta^*_{11,1} + \delta_{11,2} = 0.217 + 0.119 = 0.336$$



Dynamic multipliers VII

Although corresponding formulas to obtain the standard errors do exist, the most practical method to evaluate the uncertainty of the multipliers is by means of simulation. The procedure is straightforward:

1. Run a stochastic dynamic simulation, with drawings of both coefficients and residuals from distributions taken from the estimated results.
2. Change the exogenous variable with one unit in one period, and redo the simulation.
3. Compute the difference between the dynamic simulation paths. This is an estimate of the dynamic multipliers.
4. Redo the steps above n times.
5. Compute the standard errors or confidence intervals of the n estimates of the dynamic multipliers.

This is the approach we have used to produce the graphs above. To obtain the interim multipliers you change the exogenous variable with one unit in all periods.

References

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