Macroeconometric modelling

3 Framework

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Introduction

See Bårdsen, Hurn, and Lindsay (2004) for more details on this section.

Consider an economic model in the form of the differential equation

$$\frac{dy}{dt} = f(y, x)\ , \qquad x = x(t)\ , \tag{1}$$

$$x = \bar{x} \underset{t \to \infty}{\Rightarrow} \bar{y} \tag{2}$$

$$f(\bar{y}, \bar{x}) = 0 \tag{3}$$

Linearization I

The standard procedure is to expand about the steady-state solution

$$f(y,x) = f(\bar{y},\bar{x}) + \frac{\partial f(\bar{y},\bar{x})}{\partial y}(y-\bar{y}) + \frac{\partial f(\bar{y},\bar{x})}{\partial x}(x-\bar{x}) + R \quad (4)$$

where R is the Lagrange form of the remainder:

$$\begin{aligned} R = \frac{1}{2!}\Bigg(& \frac{\partial^2 f(\xi,\eta)}{\partial x^2}(x-\bar{x})^2 \\ & +2\frac{\partial^2 f(\xi,\eta)}{\partial x \partial y}(x-\bar{x})(y-\bar{y}) + \frac{\partial^2 f(\xi,\eta)}{\partial y^2}(y-\bar{y})^2 \Bigg) \end{aligned}$$

Linearization II

so (ξ, η) is a point such that ξ lies between y and $\bar{y}$ while η lies between x and $\bar{x}$.
Since $\bar{y}$ and $\bar{x}$ are the steady-state values for y and x respectively, the expression for $f(y, x)$ takes the simplified form

$$f(y,x) = a\,(y-\bar{y}) + \delta\,(x-\bar{x}) + R \tag{5}$$

where $a = \partial f(\bar{y}, \bar{x})/\partial y$ and $\delta = \partial f(\bar{y}, \bar{x})/\partial x$ are constants.
If f is a linear function of y and x then $R = 0$ and so

$$f(y,x) = a\left(y-\bar{y}+\frac{\delta}{a}(x-\bar{x})\right) = a\,(y-bx-c)\ , \tag{6}$$

in which $b = -\delta/a$ and $c = \bar{y} + (\delta/a)\bar{x}$.

Discretization I

Let $t_1,\ t_2,\ \cdots,\ t_k,\ \cdots$ be a sequence of times spaced h apart and let $y_1,\ y_2,\ \cdots,\ y_k,\ \cdots$ be the values of a continuous real variable $y(t)$ at these times. The backward-difference operator $\triangle$ is defined by the rule

$$\triangle y_k = y_k - y_{k-1}\ , \qquad k \geq 1\ . \tag{7}$$

By observing that $y_k = (1-\triangle)^0\, y_k$ and $y_{k-1} = (1-\triangle)^1\, y_k$, the value of y at the intermediate point $t = t_k - sh$ $(0 < s < 1)$ may be estimated by the interpolation formula

$$y(t_k - sh) = y_{k-s} = (1-\triangle)^s\, y_k\ , \qquad s \in [0,1]\ . \tag{8}$$

When s is not an integer, $(1-\triangle)^s$ should be interpreted as the power series in the backward-difference operator obtained from the

Discretization II

binomial expansion of $(1-x)^s$. This is an infinite series of differences. Specifically

$$(1-\triangle)^s = 1 - s\triangle - \frac{s(1-s)}{2!}\triangle^2 - \frac{s(1-s)(2-s)}{3!}\triangle^3 - \cdots \,. \tag{9}$$

With this preliminary background, the differential equation

$$\frac{dy}{dt} = f(y,x)\,, \qquad x = x(t)\,, \tag{10}$$

may be integrated over the time interval $[t_k, t_{k+1}]$ to obtain

$$y(t_{k+1}) - y(t_k) = \triangle y_{k+1} = \int_{t_k}^{t_{k+1}} f(y(t),x(t))\,dt \tag{11}$$

Discretization III

in which the integral on the right hand side of this equation is to be estimated by using the backward-difference interpolation formula given in equation (9). The substitution $t = t_k + sh$ is now used to change the variable of this integral from $t \in [t_k, t_{k+1}]$ to $s \in [0, 1]$. The details of this change of variable are

$$\begin{aligned}
&\int_{t_k}^{t_{k+1}} f(y(t), x(t))\, dt \\
&= \int_0^1 f(y(t_k + sh), x(t_k + sh))\,(h\, ds) \\
&= h \int_0^1 f_{k+s}\, ds
\end{aligned}$$

Discretization IV

where $f_{k+s} = f(y(t_k + sh), x(t_k + sh))$. The value of this latter integral is now computed using the interpolation formula based on (9). Thus

$$\int_0^1 f_{k+s}\, ds = \int_0^1 (1 - \triangle)^{-s} f_k\, ds$$

$$= \int_0^1 \left(f_k + s\triangle f_k + \frac{s(1+s)}{2!} \triangle^2 f_k + \frac{s(1+s)(2+s)}{3!} \triangle^3 f_k + \cdots \right) ds$$

$$= f_k + \frac{1}{2} \triangle f_k + \frac{5}{12} \triangle^2 f_k + \frac{3}{8} \triangle^3 f_k + \cdots \; .$$

Discretization V

The final form for the backward-difference approximation to the solution of this differential equation is therefore

$$\triangle y_{k+1} = hf_k + \frac{h}{2}\triangle f_k + \frac{5h}{12}\triangle^2 f_k + \frac{3h}{8}\triangle^3 f_k + \cdots \, . \tag{12}$$

Equilibrium-correction representation

The discretization scheme (12) applied to the linearized model (5), with $k = t - 1$ and $h = 1$, gives the equilibrium-correction model, EqCM, representation

$$\begin{aligned}\triangle y_t = a\,(y - bx - c)_{t-1} + R_{t-1} + \frac{1}{2}\,a(\triangle y_{t-1} - b\,\triangle x_{t-1}) + \frac{1}{2}\,\triangle R_{t-1} \\ + \frac{5}{12}\,a(\triangle^2 y_{t-1} - b\,\triangle^2 x_{t-1}) + \frac{5}{12}\,\triangle^2 R_{t-1} + \cdots\ .\end{aligned}$$

System generalization I

Consider the two-dimensional case for which $y_1 \longrightarrow \bar{y}_1$ and $y_2 \longrightarrow \bar{y}_2$ as $t \longrightarrow \infty$. Expanding with respect to y_1 and y_2 about their steady-state values yields

$$\begin{bmatrix} f_1\left(y_1, y_2\right) \\ \\ f_2\left(y_1, y_2\right) \end{bmatrix} = \begin{bmatrix} f_1\left(\bar{y}_1, \bar{y}_2\right) \\ \\ f_2\left(\bar{y}_1, \bar{y}_2\right) \end{bmatrix} + \begin{bmatrix} \dfrac{\partial f_1(\bar{y}_1, \bar{y}_2)}{\partial y_1} & \dfrac{\partial f_1(\bar{y}_1, \bar{y}_2)}{\partial y_2} \\ \dfrac{\partial f_2(\bar{y}_1, \bar{y}_2)}{\partial y_1} & \dfrac{\partial f_2(\bar{y}_1, \bar{y}_2)}{\partial y_2} \end{bmatrix} \begin{bmatrix} y_1 - \bar{y}_1 \\ \\ y_2 - \bar{y}_2 \end{bmatrix} + \begin{bmatrix} R_1 \\ \\ R_2 \end{bmatrix},$$

System generalization II

where $\left[R_1, R_2\right]'$ denotes the vector

$$\frac{1}{2!}\left[\begin{array}{c} \frac{\partial^2 f_1(\zeta,\eta)}{\partial y_1^2}\left(y_1-\bar{y}_1\right)^2+2\frac{\partial^2 f_1(\zeta,\eta)}{\partial y_1 \partial y_2}\left(y_1-\bar{y}_1\right)\left(y_2-\bar{y}_2\right)+\frac{\partial^2 f_1(\zeta,\eta)}{\partial y_2^2}\left(y_2-\bar{y}_2\right)^2 \\ \frac{\partial^2 f_2(\zeta,\eta)}{\partial y_1^2}\left(y_1-\bar{y}_1\right)^2+2\frac{\partial^2 f_2(\zeta,\eta)}{\partial y_1 \partial y_2}\left(y_1-\bar{y}_1\right)\left(y_2-\bar{y}_2\right)+\frac{\partial^2 f_2(\zeta,\eta)}{\partial y_2^2}\left(y_2-\bar{y}_2\right)^2 \end{array}\right]$$

so that

$$\left[\begin{array}{c} \frac{\partial y_1}{\partial t} \\ \frac{\partial y_2}{\partial t} \end{array}\right]=\left[\begin{array}{cc} \alpha_{11} & \alpha_{12} \\ \\ \alpha_{21} & \alpha_{22} \end{array}\right]\left[\begin{array}{c} y_1-\bar{y}_1 \\ \\ y_2-\bar{y}_2 \end{array}\right]+\left[\begin{array}{c} R_1 \\ \\ R_2 \end{array}\right].$$

System generalization III

The backward-difference approximation to the solution of the system of differential equations gives the system in EqCM form:

$$\begin{bmatrix} \triangle y_1 \\ \triangle y_2 \end{bmatrix}_t = \begin{bmatrix} -\alpha_{11} c_1 \\ -\alpha_{22} c_2 \end{bmatrix} + \begin{bmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{22} \end{bmatrix} \begin{bmatrix} y_1 - \delta_1 y_2 \\ y_2 - \delta_2 y_1 \end{bmatrix}_{t-1} + \begin{bmatrix} R_1 \\ \\ R_2 \end{bmatrix}_{t-1}$$

$$+ \frac{1}{2} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix}_{t-1} + \begin{bmatrix} \Delta R_1 \\ \\ \Delta R_2 \end{bmatrix}_{t-1}$$

$$+ \frac{5}{12} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \Delta^2 y_1 \\ \Delta^2 y_2 \end{bmatrix}_{t-1} + \begin{bmatrix} \Delta^2 R_1 \\ \\ \Delta^2 R_2 \end{bmatrix}_{t-1}$$

$$+ \frac{3}{8} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \Delta^3 y_1 \\ \Delta^3 y_2 \end{bmatrix}_{t-1} + \begin{bmatrix} \Delta^3 R_1 \\ \\ \Delta^3 R_2 \end{bmatrix}_{t-1} + \cdots .$$

with

$$\begin{aligned} c_1 &= (\bar{y}_1 + \delta_1 \bar{y}_2)\,, \quad \delta_1 = \tfrac{\alpha_{12}}{\alpha_{11}} \\ c_2 &= (\bar{y}_2 + \delta_2 \bar{y}_1)\,, \quad \delta_2 = \tfrac{\alpha_{21}}{\alpha_{22}} \end{aligned}$$

Comments I

1. Econometric specification implies truncation: Diagnostic testing
2. Various forms of steady state are possible:
    2.1 Filter the data to remove trends, hopefully achieving stationarity;
    2.2 Impose the theoretical balanced growth path of the model on the data;.
    2.3 Estimate the balanced growth paths: finding the number of common trends and identifying and estimating cointegrating relationships.

Cointegration example I

Assume the logs of the real wage rw_t and productivity z_t to be integrated of order one, but cointegrated, so

$$rw_t \sim I(1), \; \Delta rw_t \sim I(0) \tag{13}$$

$$z_t \sim I(1), \; \Delta z_t \sim I(0) \tag{14}$$

$$(rw - \beta z)_t \sim I(0). \tag{15}$$

(Constant unemployment)

Cointegration example II

Letting $y_{1t} \equiv (rw - \beta z)_t$ and $y_{2t} \equiv \Delta z_t$, assuming linearity, so $R_i = 0$, and ignoring higher order dynamics for ease of exposition, then gives:

$$\begin{aligned} \triangle rw_t &= -\alpha_{11} c_1 + \alpha_{11}(rw - \beta z)_{t-1} + \beta \Delta z_t - \alpha_{12} \Delta z_{t-1} \\ \Delta z_t &= -\alpha_{22} \left(\bar{z} + \frac{\alpha_{21}}{\alpha_{22}} \overline{rw} \right) + (\alpha_{22} - 1)\ \Delta z_{t-1} - \alpha_{21}(rw - \beta z)_{t-1} \end{aligned}$$

So if $\alpha_{21} = 0$ and $|\alpha_{22} - 1| < 1$ the system simplifies to the familiar expositon of a bivariate cointegrated system with z being weakly exogenous for β:

$$\begin{aligned} \triangle rw_t &= -\alpha_{11} c_1 + \alpha_{11}(rw - \beta z)_{t-1} + \beta \Delta z_t - \alpha_{12} \Delta z_{t-1} \\ \Delta z_t &= -\alpha_{22}\, \bar{z} + (\alpha_{22} - 1)\ \Delta z_{t-1}. \end{aligned}$$

Looking ahead I

From a discretized and linearized cointegrated VAR representation to a dynamic Simultaneous Model (SEM) in three steps

1. Linearized and discretized approximation as a data-coherent statistical system representation in the form of a cointegrated VAR

$$\Delta \mathbf{y}_t = \nu + \Pi \mathbf{y}_{t-1} + \sum_{i=1}^{k} \Gamma_i \Delta \mathbf{y}_{t-i} + \mathbf{u}_t, \tag{16}$$

2. Identify the steady state, by testing and imposing overidentifying restrictions on the cointegration space:

$$\Delta \mathbf{y}_t = \nu + \alpha^* \beta^{*\prime} \mathbf{y}_{t-1} + \sum_{i=1}^{k} \Gamma_i \Delta \mathbf{y}_{t-i} + \mathbf{u}_t,$$

Looking ahead II

From a discretized and linearized cointegrated VAR representation to a dynamic Simultaneous Model (SEM) in three steps

3. Identify the dynamics, by testing and imposing overidentifying restrictions on the dynamics:

$$\mathbf{A}_0 \Delta \mathbf{y}_t = \mathbf{A}_0 \nu + \mathbf{A}_0 \alpha^* \beta^{*'} \mathbf{y}_{t-1} + \sum_{i=1}^{k} \mathbf{A}_0 \Gamma_{t-i} \Delta \mathbf{y}_{t-i} + \mathbf{A}_0 \mathbf{u}_t.$$

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