Macroeconometric modelling

2 Background

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Models with steady state

We are interested in models with a steady state
They need not be long-run growth models, but they need to be stable so

$$\lim_{t \to \infty} f\left(Z_t\right) = f\left(\bar{Z}\right),$$

where Z is a vector of suitably transformed stationary variables.
Otherwise we cannot do policy analysis. This is Samuelsons (1941, 1942) correspondence principle—see Evans and Honkapohja (2007) for a recent reappraisal.
To motivate, some classical and simple model examples follow.

Keynesian multiplier model

Foundation of policy and disequilibrium macro

No growth, discrete time:

$$Y_t = C_t + \bar{I}$$
$$C_t = c_0 + c_1 Y_{t-1}$$

with constant steady state

$$\bar{Y} = \frac{c_0}{1-c_1} + \frac{1}{1-c_1}\bar{I}.$$

if $c_1 < 1$. The model is cast in Equilibrium Correction form (EqCM) as:

$$\begin{aligned} Y_t - Y_{t-1} &= c_0 - (1-c_1) Y_{t-1} + \bar{I} \\ &= -(1-c_1)\left[Y_{t-1} - \left(\underbrace{\frac{c_0}{1-c_1} + \frac{1}{1-c_1}\bar{I}}_{\bar{Y}} \right) \right] \\ \Delta Y_t &= -(1-c_1)\left(Y_{t-1} - \bar{Y}\right). \end{aligned}$$

Solow-Swan growth model I

Foundation of growth literature

In discrete time, with population growth and technological progress, see f. ex. Blanchard and Fischer (1989):

$$Y_t = C_t + I_t \tag{1}$$

$$Y_t = F\left(A_t N_t, K_{t-1}\right) \tag{2}$$

$$C_t = (1-s)\, Y_t \tag{3}$$

$$I_t = K_t - K_{t-1} + \delta K_{t-1} \tag{4}$$

$$\left.\begin{array}{l} \frac{\Delta A_t}{A_{t-1}} = a \\ \frac{\Delta N_t}{N_{t-1}} = n \end{array}\right\} \frac{\Delta \left(AN\right)_t}{\left(AN\right)_{t-1}} = (1+a)(1+n) - 1 = g \tag{5}$$

If the production function satisfies the Inada conditions, the stability condition $\frac{(1-\delta)}{1+g} < 1$ of

$$s \times f\left(\frac{K_{t-1}}{(AN)_t}\right) = (1+g)\left(\frac{K_t}{(AN)_{t+1}}\right) - (1-\delta)\left(\frac{K_{t-1}}{(AN)_t}\right) \tag{6}$$

Solow-Swan growth model II

Foundation of growth literature

gives per capita efficiency level of capital of

$$\lim_{t\to\infty} \frac{K_{t-1}}{(AN)_t} = \frac{g+\delta}{s}.$$

Ramsey growth model in continous time

Foundation of the RBC literature

1. order Taylor expansion of model in per capita terms, no technological progress or depreciation, see Blanchard and Fischer (1989):

$$\dot{c}(t) = -\beta (k - k^*)$$
$$\dot{k}(t) = -(c - c^*) + \theta (k - k^*)$$

The stability conditon is now $\frac{\theta - \sqrt{(4\beta + \theta^2)}}{2} < 0$.

Phillips' (1954, 1957) proportional control model

Foundation of policy control literature

Continous time with no growth, see Turnovsky (1977, p. 322-323):

$$\begin{pmatrix} \dot{Y}(t) \\ \dot{G}(t) \end{pmatrix} = \begin{pmatrix} \alpha(c-1) & \alpha \\ -\beta\gamma & -\beta \end{pmatrix} \begin{pmatrix} Y - Y^* \\ G - G^* \end{pmatrix}$$

The stability conditions are

$$\alpha(c-1) - \beta < 0$$
$$\alpha\beta(\gamma + 1 - c) > 0,$$

New Keynesian Model with Taylor rule

Presently

Discrete time, no growth, see Wickens (2008, p. 358-369):

$$
\begin{aligned}
\Delta p_t &= \mu + \beta E_t \Delta p_{t+1} + \gamma x_t + e_{\pi t} \\
x_t &= E_t x_{t+1} - \alpha \left(R_t - E_t \Delta p_{t+1} - \theta \right) + e_{xt} \\
R_t &= \theta + \pi^* + \mu \left(\Delta p_t - \Delta p^* \right) + \nu x_t + e_{Rt}
\end{aligned}
$$

where Δp is inflation, x is the output gap and R is the interest rate and the e-s are shocks. If $1 > \frac{\beta}{1+\alpha(\nu+\mu\gamma)} > 0$, the solutions for inflation and output gap are

$$
\begin{aligned}
\Delta p_t &= \Delta p^* + \frac{\left(1 + \alpha\nu\right) e_{\pi t} + \gamma \left(e_{xt} - \alpha e_{Rt} \right)}{1 + \alpha \left(\nu + \mu\gamma \right)} \\
x_t &= \frac{-\alpha\mu e_{\pi t} - e_{xt} + \alpha e_{Rt}}{1 + \alpha \left(\nu + \mu\gamma \right)}
\end{aligned}
$$

so on average inflation Δp is equal to the target Δp^* and the output gap x is zero.

Vector Autoregressive Representations (VARs) I

We follow the exposition in Lütkepohl (2005)—see there for further details

We are interested in analyzing the k-dimensional VAR(p) process $\mathbf{y}_t$

$$\mathbf{y}_t = \sum_{i=1}^{p} A_i \mathbf{y}_{t-i} + \mathbf{u}_t, \ \mathbf{u}_t \sim (\mathbf{0}, \Sigma_u)$$

The system is stable if

$$|-\mathbf{\Pi}| = \det\left(\mathbf{I}_k - \mathbf{A}_1 z - \cdots - \mathbf{A}_p z^p\right) \neq 0 \text{ for } |z| \leq 1.$$

Structural VARs (SVARs) I

The system has a so-called Wold Moving Average representation (intuitively, "solving" by recursive substitution):

$$\mathbf{y}_t = \sum_{i=0}^{\infty} \Phi_i \mathbf{u}_{t-i}, \ \Phi_s = \sum_{j=1}^{s} \Phi_{s-j} A_j, \ s = 1, \ 2, \ldots, \ \Phi_o = \mathbf{I}_k$$

- The elements of Φ_j are the responses of the variables to changes in the errors.
- Since the errors most likely are correlated, the responses will not reflect shocks to the variables.
- Solution:
  - make errors uncorrelated: innovations
  - the new response functions should now reflect unique impulses to the variables: impulse response functions.
  - Example: the "**A**-model"

Structural VARs (SVARs) II

1. Assume the underlying identified model

$$\mathbf{A}_0\mathbf{y}_t = \mathbf{A}_0\sum_{i=1}^{p} A_i\mathbf{y}_{t-i} + \mathbf{A}_0\mathbf{u}_t.$$

$$\mathbf{A}_0\mathbf{u}_t = \varepsilon_t \sim (\mathbf{0}, \Sigma_\varepsilon), \Sigma_\varepsilon = \mathbf{A}_0\Sigma_u\mathbf{A}_0'$$

   where $\Sigma_\varepsilon = \mathbf{A}_0\Sigma_u\mathbf{A}_0'$ is the diagonal covariance matrix of the model errors—the innovations.

2. The MA-representation is now

$$\mathbf{y}_t = \sum_{i=0}^{\infty}\Phi_i\mathbf{A}_0^{-1}\mathbf{A}_0\mathbf{u}_{t-i} = \sum_{i=0}^{\infty}\Theta_i\varepsilon_{t-i}$$

3. The elements of Θ_i are the unique impulse responses.

- We will return to this under the heading of dynamic multipliers.
- The fundamental paper on SVARs with common trends and cointegration is King, Plosser, Stock, and Watson (1991).

Structural VARs (SVARs) III

- See Anders Warne's lecture notes at http://www.texlips.net/download/lecture_notes.pdf for a brilliant introduction to this literature.

VAR representations of DSGE models I

VAR solution

This follows Fernandez-Villaverde, Rubio-Ramirez, Sargent, and Watson (2007). In compact notation the (log-linearised) equilibrium conditions of a large class of models, including DSGE models, can be written

$$FE_t\xi_{t+1} + G\xi_t + H\xi_{t-1} = Ju_t, \qquad (7)$$

where ξ_t is a vector of state variables and u_t is a vector of uncorrelated white noise shocks (e.g., shocks to technology and preferences) and the elements in the coefficient matrices are non-linear functions of the underlying structural parameters in the DSGE model.

If the Blanchard-Kahn conditions (see Blanchard and Kahn (1980)) are satisfied, the model has a unique stable solution

$$\xi_t = A\xi_{t-1} + Bu_t.$$

VAR representations of DSGE models II

VAR solution

Adding a set of measurement equations relating the elements of ξ_t to a vector of observable variables y_t gives the state-space representation

$$\xi_t = A\xi_{t-1} + Bu_t \tag{8}$$
$$y_t = C\xi_{t-1} + Du_t. \tag{9}$$

If D is non-singular we get from (9)

$$u_t = D^{-1}\left(y_t - C\xi_{t-1}\right),$$

which substituted into (8) and rearranging gives

$$\begin{aligned}\xi_t &= A\xi_{t-1} + BD^{-1}\left(y_t - C\xi_{t-1}\right)\\ &= \left(A - BD^{-1}C\right)\xi_{t-1} + BD^{-1}y_t\\ \left(I - \left(A - BD^{-1}C\right)L\right)\xi_t &= BD^{-1}y_t\end{aligned}$$

VAR representations of DSGE models III

VAR solution

- The solution can be approximated by a vector autoregression (VAR) (or a vector equilibrium correction model (VEqCM)) if the moving average (MA) representation is invertible (that is, it must be possible to recover the shocks u_t from the current and lagged values of the observables (see e.g., Watson (1994)).
- If the number of observables equals the number of shocks and D^{-1} exists, a necessary and sufficient condition for invertibility is that the eigenvalues of $A - BD^{-1}C$ are strictly less than one in modulus (see Fernandez-Villaverde, Rubio-Ramirez, Sargent, and Watson (2007)).

VAR representations of DSGE models IV

VAR solution

If this condition is satisfied, then

$$\xi_t = \sum_{j=0}^{\infty} \left(A - BD^{-1}C\right)^j BD^{-1} y_{t-j}.$$

Shifting back one period and substituting into (9) then gives the VAR representation

$$y_t = C \sum_{j=0}^{\infty} \left(A - BD^{-1}C\right)^j BD^{-1} y_{t-j-1} + Du_t. \tag{10}$$

- If all the endogenous state variables are observable and included in the VAR, the VAR representation is of finite order (e.g., Ravenna (2007)).
- In general, however, the VAR is of infinite order (corresponding to a VARMA representation).

An example: the New Keynesian Phillips Curve

The model is

$$\Delta p_t = b^f_{p1} E_t \Delta p_{t+1} + b^b_{p1} \Delta p_{t-1} + b_{p2} x_t + \varepsilon_{pt} \tag{11}$$

$$x_t = b_x x_{t-1} + \varepsilon_{xt} \tag{12}$$

where all coefficients are assumed to be between zero and one. The solution (the statistical system) is

$$\begin{pmatrix} \Delta p \\ x \end{pmatrix}_t = \begin{pmatrix} \alpha_1 & \frac{b_{p2} b_x}{b^f_{p1}(\alpha_2 - b_x)} \\ 0 & b_x \end{pmatrix} \begin{pmatrix} \Delta p \\ x \end{pmatrix}_{t-1} + \begin{pmatrix} u_p \\ u_x \end{pmatrix}_t,$$

$$\begin{pmatrix} u_p \\ u_x \end{pmatrix}_t = \begin{pmatrix} \frac{1}{b^f_{p1}\alpha_2} & \frac{b_{p2}}{b^f_{p1}(\alpha_2 - b_x)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_p \\ \varepsilon_x \end{pmatrix}_t$$

Derivation of the solution

Hard to find I

Standard dynamic price-wage model

$$\begin{pmatrix} 1 & -\gamma_2 \\ -\delta_2 & 1 \end{pmatrix} \begin{pmatrix} \Delta p \\ x \end{pmatrix}_t = \begin{pmatrix} 1 & \gamma_1 \\ \delta_1 & 1 \end{pmatrix} \begin{pmatrix} \Delta p \\ x \end{pmatrix}_{t-1} + \begin{pmatrix} \gamma_3 & 0 \\ 0 & \delta_3 \end{pmatrix} \begin{pmatrix} gap \\ u \end{pmatrix}_t + \begin{pmatrix} e_p \\ e_x \end{pmatrix}_t$$

with statistical system (reduced form)

$$\begin{pmatrix} \Delta p \\ x \end{pmatrix}_t = \begin{pmatrix} -\frac{\gamma_2 \delta_1 + 1}{\gamma_2 \delta_2 - 1} & -\frac{\gamma_1 + \gamma_2}{\gamma_2 \delta_2 - 1} \\ -\frac{\delta_1 + \delta_2}{\gamma_2 \delta_2 - 1} & -\frac{\gamma_1 \delta_2 + 1}{\gamma_2 \delta_2 - 1} \end{pmatrix} \begin{pmatrix} \Delta p \\ x \end{pmatrix}_{t-1} + \begin{pmatrix} v_p \\ v_x \end{pmatrix}_t ,$$

$$\begin{pmatrix} v_p \\ v_x \end{pmatrix}_t = \begin{pmatrix} -\frac{\gamma_3}{\gamma_2 \delta_2 - 1} & -\gamma_2 \frac{\delta_3}{\gamma_2 \delta_2 - 1} \\ -\gamma_3 \frac{\delta_2}{\gamma_2 \delta_2 - 1} & -\frac{\delta_3}{\gamma_2 \delta_2 - 1} \end{pmatrix} \begin{pmatrix} gap \\ u \end{pmatrix}_t + \begin{pmatrix} -\frac{e_p + \gamma_2 e_x}{\gamma_2 \delta_2 - 1} \\ -\frac{e_x + \delta_2 e_p}{\gamma_2 \delta_2 - 1} \end{pmatrix}_t$$

Hard to find II

could be observationally equivalent–same with conditional models.

- See Bårdsen, Jansen, and Nymoen (2004) for testing with conditional models.

Estimation and identification I

Errors in variables

Remember that the model is

$$\Delta p_t = b_{p1}^f E_t \Delta p_{t+1} + b_{p1}^b \Delta p_{t-1} + b_{p2} x_t + \varepsilon_{pt}$$

which can be rewritten as

$$\pi = \gamma E_t \pi_{t+1} + \delta x_t + v_{pt}.$$

where $\pi_t = \Delta p_t - \alpha_1 \Delta p_{t-1}$ and α_1 is the backward stable root of the solution. The model is often estimated by means of instrumental variables, using the "errors in variables" method (evm)—where expected values are replaced by actual values and the expectational errors:

$$\pi_t = \gamma \pi_{t+1} + \delta x_t + v_{pt} - \gamma \eta_{t+1}. \tag{13}$$

Estimation and identification II

Errors in variables

The implications of estimating the model by means of the "errors in variables" method is to induce moving average errors. Following Blake (1991), this can be readily seen using the expectational errors as follows.

Estimation and identification III

Errors in variables

1. Lead (18) one period and subtract the expectation to find the RE error:

$$
\begin{aligned}
\eta_{t+1} &= \gamma E_t \pi_{t+2} + \delta x_{t+1} + v_{pt+1} - E_t \pi_{t+1} \\
&= \gamma \left(\frac{\delta b_x}{1 - \gamma b_x} \right) x_{t+1} + \delta x_{t+1} + v_{pt+1} - \left(\frac{\delta b_x}{1 - \gamma b_x} \right) x_t \\
&= \left(\frac{\delta}{1 - \gamma b_x} \right) (x_{t+1} - b_x x_t) + v_{pt+1} \\
&= \left(\frac{\delta}{1 - \gamma b_x} \right) \varepsilon_{xt+1} + v_{pt+1}
\end{aligned}
$$

Estimation and identification IV

Errors in variables

2. Substitute into (13):

$$\pi_t = \gamma \pi_{t+1} + \delta x_t + v_{pt} - \gamma v_{pt+1} - \left(\frac{\gamma \delta}{1 - \gamma b_x} \right) \varepsilon_{xt+1}.$$

3. Finally, reexpress in terms of original variables:

$$\Delta p_t - \alpha_1 \Delta p_{t-1} = \left(\frac{1}{\alpha_2} \right) (\Delta p_{t+1} - \alpha_1 \Delta p_t) + \left(\frac{b_{p2}}{b^f_{p1} \alpha_2} \right) x_t$$
$$+ \left(\frac{1}{b^f_{p1} \alpha_2} \right) \varepsilon_{pt} - \left(\frac{1}{\alpha_2} \right) \left(\frac{1}{b^f_{p1} \alpha_2} \right) \varepsilon_{pt+1} - \left(\frac{\left(\frac{1}{\alpha_2} \right) \left(\frac{b_{p2}}{b^f_{p1} \alpha_2} \right)}{1 - \left(\frac{1}{\alpha_2} \right) b_x} \right) \varepsilon_{xt+1}.$$

Estimation and identification V

Errors in variables

Giving

$$\begin{aligned}\Delta p_t = b_{p1}^f \Delta p_{t+1} + b_{p1}^b \Delta p_{t-1} + b_{p2} x_t \\ + \varepsilon_{pt} - \left(\frac{1}{\alpha_2}\right) \varepsilon_{pt+1} - \left(\frac{b_{p2}}{\alpha_2 - b_x}\right) \varepsilon_{xt+1},\end{aligned}$$

where we have exploited the two well-known relationships between the roots:

$$\begin{aligned}\alpha_1 + \alpha_2 &= \frac{1}{b_{p1}^f} \\ \alpha_1 \alpha_2 &= \frac{b_{p1}^b}{b_{p1}^f}.\end{aligned}$$

Estimation and identification VI

Errors in variables

So even though the original model has white noise errors, the estimated model will have first order moving average errors.

Does the MA(1) process prove that the forward solution applies? I

Assume that the true model is

$$\Delta p_t = b_{p1}\Delta p_{t-1} + \varepsilon_{pt}, \quad |b_{p1}| < 1$$

and the the following model is estimated by means of instrumental variables

$$\Delta p_t = b^f_{p1}\Delta p_{t+1} + \varepsilon^f_{pt}$$

What are the proprties of ε^f_{pt}?

$$\varepsilon^f_{pt} = \Delta p_t - b^f_{p1}\Delta p_{t+1}$$

Assume, as is common in the literature, that we find that $b^f_{p1} \approx 1$. Then

$$\begin{aligned}\varepsilon^f_{pt} &\approx \Delta p_t - \Delta p_{t+1} = -\Delta^2 p_{t+1} \\ &= -\left[\varepsilon_{pt+1} + (b_{p1} - 1)\varepsilon_{pt} + ...\right].\end{aligned}$$

Does the MA(1) process prove that the forward solution applies? II

So we get a model with a moving average residual, but this time the reason is not forward looking behaviour but misspecification.

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Appendix: Solution of the New Keynesian Phillips Curve

Method of repeated substitution

All of the Rational expectations techniques rely on the law of iterated expectations, saying

$$E_t E_{t+k} x_{t+j} = E_t x_{t+j},$$

that your average revisions of expectations given more information will be zero. The method of repeated substitution is the brute force solution, cumbersome and not very general. But since it's also instructive to see exactly what goes on, we use with this method. See Bårdsen, Eitrheim, Jansen, and Nymoen (2005, Appendix A1) for alternative analytical methods and McCandless (2008) and Dejong and Dave (2007) for general methods based on matrix decompositions.

We start by using a trick to get rid of the lagged dependent variable, following Pesaran (1987, p. 108-9), by defining

$$\Delta p_t = \pi_t + \alpha \Delta p_{t-1} \tag{14}$$

where α will turn out to be the backward stable root of the process of Δp_t.

We take expectations one period ahead

$$\begin{aligned} E_t \Delta p_{t+1} &= E_t \pi_{t+1} + \alpha E_t \Delta p_t \\ E_t \Delta p_{t+1} &= E_t \pi_{t+1} + \alpha \pi_t + \alpha^2 \Delta p_{t-1}. \end{aligned}$$

Next, we substitute for $E_t \Delta p_{t+1}$ into original model:

$$\pi_t + \alpha \Delta p_{t-1} = b^f_{p1} \left(E_t \pi_{t+1} + \alpha \pi_t + \alpha^2_{t-1} \Delta p_{t-1} \right)$$
$$+ b^b_{p1} \Delta p_{t-1} + b_{p2} x_t + \varepsilon_{pt}$$
$$\pi_t = \left(\frac{b^f_{p1}}{1 - b^f_{p1} \alpha} \right) E_t \pi_{t+1} + \left(\frac{b^f_{p1} \alpha^2 - \alpha + b^b_{p1}}{1 - b^f_{p1} \alpha} \right) \Delta p_{t-1}$$
$$+ \left(\frac{b_{p2}}{1 - b^f_{p1} \alpha} \right) x_t + \left(\frac{1}{1 - b^f_{p1} \alpha} \right) \varepsilon_t .$$

The parameter α is defined by

$$b^f_{p1} \alpha^2 - \alpha + b^b_{p1} = 0$$

or

$$\alpha^2 - \frac{1}{b^f_{p1}} \alpha + \frac{b^b_{p1}}{b^f_{p1}} = 0 \tag{15}$$

with the solutions

$$\left.\begin{array}{c}\alpha_1\\ \alpha_2\end{array}\right\} = \frac{1 \pm \sqrt{1 - 4b^f_{p1} b^b_{p1}}}{2b^f_{p1}}. \tag{16}$$

The model will typically have a saddle point behaviour with one root bigger than one and one smaller than one in absolute value. In the following we will use the backward stable solution, defined by:

$$\left|\alpha_1 = \frac{1 - \sqrt{1 - 4b^f_{p1} b^b_{p1}}}{2b^f_{p1}}\right| < 1.$$

In passing might be noted that the restriction $b_{p1}^{b} = 1 - b_{p1}^{f}$ often imposed in the literature implies the roots

$$\alpha_1 = \frac{1 - b_{p1}^{f}}{b_{p1}^{f}}$$
$$\alpha_2 = 1,$$

as given in (16) as before. We choose $|\alpha_1| < 1$ in the following. So we now have a pure forward-looking model

$$\pi_t = \left(\frac{b_{p1}^{f}}{1 - b_{p1}^{f}\alpha_1} \right) E_t \pi_{t+1} + \left(\frac{b_{p2}}{1 - b_{p1}^{f}\alpha_1} \right) x_t + \left(\frac{1}{1 - b_{p1}^{f}\alpha_1} \right) \varepsilon_{pt}.$$

Finally, using the relationship

$$\alpha_1 + \alpha_2 = \frac{1}{b^f_{p1}}$$

between the roots, see f.ex. Chiang (1984, p. 506), so:

$$1 - b^f_{p1}\alpha_1 = b^f_{p1}\alpha_2, \tag{17}$$

the model becomes

$$\pi_t = \left(\frac{1}{\alpha_2}\right) E_t\pi_{t+1} + \left(\frac{b_{p2}}{b^f_{p1}\alpha_2}\right) x_t + \left(\frac{1}{b^f_{p1}\alpha_2}\right) \varepsilon_{pt} \tag{18}$$

$$\pi = \gamma E_t\pi_{t+1} + \delta x_t + v_{pt} \tag{19}$$

Following Davidson (2000, p. 109–10), we now derive the solution in two steps:

1. Find $E_t\pi_{t+1}$
2. Solve for π_t.

Find $E_t\pi_{t+1}$:
Define the expectational errors as:

$$\eta_{t+1} = \pi_{t+1} - E_t\pi_{t+1}. \tag{20}$$

We start by reducing the model to a single equation:

$$\pi_t = \gamma\pi_{t+1} + \delta b_x x_{t-1} + \delta\varepsilon_{xt} + v_{pt} - \gamma\eta_{t+1}.$$

Solving forwards then produces:

$$\begin{aligned}
\pi_t &= \gamma\left(\gamma\pi_{t+2} + \delta b_x x_t + \delta\varepsilon_{xt+1} + v_{pt+1} - \gamma\eta_{t+2}\right) \\
&\quad + \delta b_x x_{t-1} + \delta\varepsilon_{xt} + v_{pt} - \gamma\eta_{t+1} \\
&= \left(\delta b_x x_{t-1} + \delta\varepsilon_{xt} + v_{pt} - \gamma\eta_{t+1}\right) \\
&\quad + \gamma\left(\delta b_x x_t + \delta\varepsilon_{xt+1} + v_{pt+1} - \gamma\eta_{t+2}\right) + (\gamma)^2\,\pi_{t+2} \\
&= \sum_{j=0}^{n} (\gamma)^j\left(\delta b_x x_{t+j-1} + \delta\varepsilon_{xt+j} + v_{pt+j} - \gamma\eta_{t+j+1}\right) + (\gamma)^{n+1}\,\pi_{t+n+1}.
\end{aligned}$$

By imposing the transversality condition:

$$\lim_{n\to\infty} (\gamma)^{n+1}\,\pi_{t+n+1} = 0$$

and then taking expectations conditional at time t, we get the "discounted solution":

$$E_t \pi_{t+1} = \sum_{j=0}^{\infty} (\gamma)^j \left(\delta b_x E_t x_{t+j} + \delta E_t \varepsilon_{xt+j+1} + E_t v_{pt+j+1} - \gamma E_t \eta_{t+j+2}\right)$$

$$= \sum_{j=0}^{\infty} (\gamma)^j \left(\delta b_x E_t x_{t+j}\right).$$

However, we know the process for the forcing variable, so:

$$\begin{aligned}
E_{t-1} x_t &= b_x x_{t-1} \\
E_t x_t &= x_t \\
E_t x_{t+1} &= b_x x_t \\
E_t x_{t+2} &= E_t \left(E_{t+1} x_{t+2}\right) = E_t b_x x_{t+1} = b_x^2 x_t \\
E_t x_{t+j} &= b_x^j x_t.
\end{aligned}$$

We can therefore substitute in:

$$\begin{aligned} E_t\pi_{t+1} &= \sum_{j=0}^{\infty} (\gamma)^j \left(\delta b_x b_x^j x_t\right) \\ &= \delta b_x \sum_{j=0}^{\infty} (\gamma b_x)^j \, x_t \\ &= \left(\frac{\delta b_x}{1-\gamma b_x}\right) x_t. \end{aligned}$$

and substitute back the expectation into the original equation:

$$\begin{aligned} \pi_t &= \gamma E_t \pi_{t+1} + \delta x_t + v_{pt} \\ &= \gamma \left(\frac{\delta b_x}{1-\gamma b_x}\right) x_t + \delta x_t + v_{pt}. \end{aligned}$$

Finally, using (14) and (19) we get the complete solution:

$$\Delta p_t - \alpha_1 \Delta p_{t-1} = \left(\frac{b_{p1}^f}{b_{p1}^f \alpha_2}\right) \left(\frac{\left(\frac{b_{p2}}{b_{p1}^f \alpha_2}\right) b_x}{1 - \left(\frac{b_{p1}^f}{b_{p1}^f \alpha_2}\right) b_x}\right) x_t + \left(\frac{b_{p2}}{b_{p1}^f \alpha_2}\right) x_t + \left(\frac{1}{b_{p1}^f \alpha_2}\right) \varepsilon_{pt} \tag{21}$$

$$= \left(\frac{1}{\alpha_2}\right) \left(\frac{b_{p2} b_x}{b_{p1}^f (\alpha_2 - b_x)}\right) x_t + \left(\frac{b_{p2}}{b_{p1}^f \alpha_2}\right) x_t + \left(\frac{1}{b_{p1}^f \alpha_2}\right) \varepsilon_{pt}$$

$$\Delta p_t = \alpha_1 \Delta p_{t-1} + \left(\frac{b_{p2}}{b_{p1}^f (\alpha_2 - b_x)}\right) x_t + \left(\frac{1}{b_{p1}^f \alpha_2}\right) \varepsilon_{pt} \tag{22}$$

Return