The Savings Multiplier[*]

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Abstract

We develop a theory of macroeconomic development based on the novel concept of *savings multiplier*: capital accumulation changes relative prices and income shares between generations, creating further incentives to accumulate and thereby rising saving rates as the economy develops. The savings multiplier is a feedback effect of growth on saving rates that hinges on two mechanisms. First, accumulation raises wages and leads to redistribution from the consuming old to the saving young. Second, higher wages raise the price of services consumed by the old, and the anticipation of such price rise prompts the young to increase their savings. Our theory captures important aspects of China's development, including the causes of rising saving rates in the past and the possible consequences of future welfare reforms.

Keywords: Overlapping generations, Growth, Savings.
JEL: O11, D91, E21

1 Introduction

This paper presents a theory of macroeconomic development based on the novel concept of *savings multiplier*: capital accumulation sparks output growth but also induces changes in relative prices and intergenerational income shares that create further incentives to accumulate, implying rising saving rates as the economy develops. The savings multiplier creates a feedback effect of growth on savings that magnifies the impact of exogenous shocks – namely, demographic change, policy reforms, productivity shocks – on capital per capita in the long run. The scope of our results is twofold. First, the savings multiplier introduces circular causality in the savings-growth relationship and thus provides a new explanation for rising saving rates in developing countries. Second, our theory captures important aspects of China's economic performance: our model suggests what type of forces have fuelled households saving rates in the past, and what could be the future consequences of the recently announced welfare reforms. We discuss each point in turn below.

Rising saving rates characterized the growth process of most developed economies. Lewis (1954) provides an early recognition of this stylized fact, stressing the potential role played by feedback effects:

> The central problem in the theory of economic development is to understand the process by which a community which was previously saving and investing 4 or 5 per cent of its national income or less, converts itself into an economy where voluntary saving is running at about 12 to 15 per cent of national income or more. [...] We cannot explain any "industrial" revolution [...] until we can explain why saving increased relatively to national income. (Lewis, 1954: p.155).

The issue of causality in the relationship between growth and saving rates is still an open question (see Deaton, 2010). Standard growth theories tell us that saving rates drive development during the transition (Solow, 1956) if not forever (Romer, 1989). Empirical evidence however suggests that causality may run in the opposite direction (Carroll and Weil, 1994; Attanasio et al. 2000; Rodrik, 2000). The topic received attention in the growth literature of the late 1990s – mostly dedicated to the stunning performance of East Asian economies – but, somewhat surprisingly, only a few contributions attempted at developing new theories to explain the effects of growth on saving rates. A prominent example is the theory of Relative Consumption, where households' utility depends on current consumption relative to a benchmark level which may reflect habit formation (Carroll et al. 2000), Catching-Up with the Joneses phenomena (Alvarez-Cuadrado et al. 2004), or international status seeking (Valente, 2009). In Relative Consumption models, feedback effects arise because economic growth raises the benchmark consumption level over time and the agents' willingness to catch-up with the benchmark prompts households to adjust savings accordingly.[1] Our theory of the savings multiplier is different be-

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[1] Carroll et al. (2000) argue that models with habit formation (i.e., the benchmark is determined by past

cause the feedback effects of growth on saving rates hinge, in our model, on the economy's demographic structure – which comprises overlapping generations of agents with finite lives – and on the allocation of labor between different production sectors – namely, a manufacturing sector and a labor-intensive sector supplying services that agents demand in their second period of life (e.g., old-age care).

In our model, the first channel through which growth affects saving rates is what we term the *intergenerational distribution effect*. Higher savings imply both higher capital stock and increased demand for care by the old, both fueling wage increases. The income distribution shifts in favor of the wage earners – that is, accumulation raises the income share of savers relative to the old agents – which stimulates further savings and capital accumulation. The second channel is what we term the *old-age requirement effect*. Increased savings and capital accumulation push the anticipated future wage up, making old-age care more expensive. To compensate for the increased future costs of care, young agents increase their savings relative to current income. This gives an additional channel whereby savings and capital accumulation stimulate further savings and capital accumulation. During the transition to the long-run equilibrium, savings rates increase over time, the share of employment in the manufacturing sector declines, the income distribution shifts in favor of the young, and an increasing share of private expenditures is allocated to the purchase of old-age care services. This mechanism clearly distinguishes our notion of savings multiplier, which operates on the supply side under full employment conditions, from the traditional concept of demand multiplier according to which income is pushed up from the side of demand when factors of production are not fully utilized. To our knowledge, neither the term 'savings multiplier' nor its underlying concept have been previously introduced in the literature.

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Although our contribution is theoretical, the key motivation of our analysis lies in the empirical literature on Asian economies, and on the experience of China in particular. Since 1978, real per capita GDP in China has increased tenfold, and fast output growth was accompanied by massive capital accumulation. After drastic policy changes in the late 1970s, savings and investment as a share of GDP increased sharply. Importantly, savings and investment rates continued to grow thereafter: graph (a) in Figure 1 shows that more than 40% of GDP has been invested, while more than 50% of GDP has been saved, over the last years.

China's saving behavior inspired a huge body of empirical literature but there is a lack of new theories that could explain the most puzzling fact, namely, that households have increased their

consumption levels) consistently match the behavior of saving rates in Japan. Alvarez-Cuadrado (2008) uses the 'Catching-Up with the Joneses' variant (i.e., the benchmark is determined by the economy's average consumption level) to reproduce the dynamics of saving rates in post-war Europe during the reconstruction period. Valente (2009) assumes international status seeking (i.e., the benchmark is determined by a foreign economy's average consumption level) to match saving rates differentials and terms of trade between US and Singapore during the 1960-2010 period.
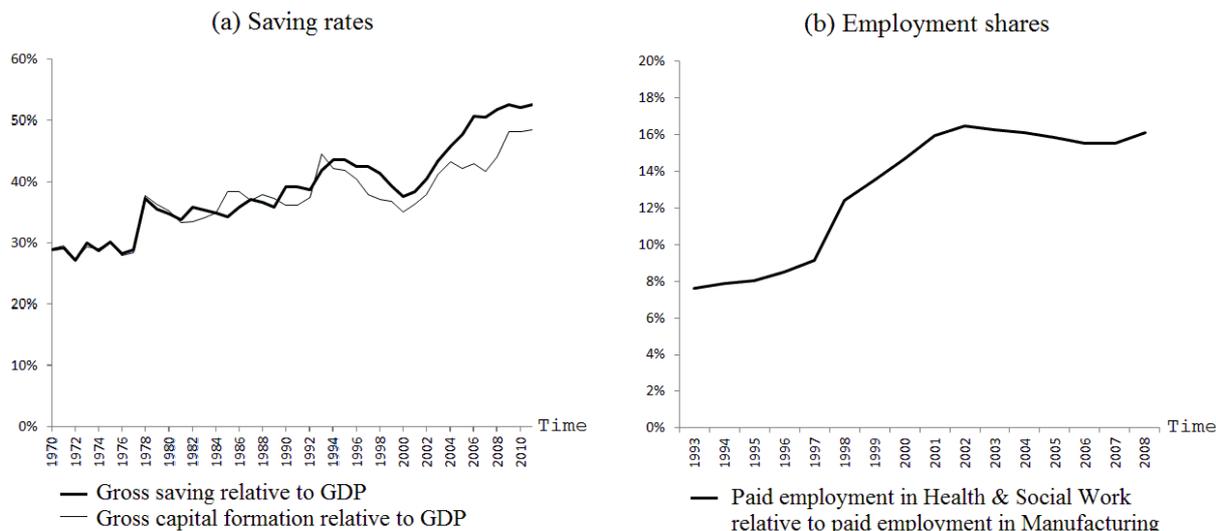
Figure 1: Graph (a): saving and investment shares of GDP in China 1970-2010 (source: World Bank). Graph (b): paid employment in Health and Social Work relative to paid employment in Manufacturing in China 1993-2008 (source: authors calculations on LABORSTA Table 2E, International Labor Organization).

savings rate, despite being quite poor, having fast income growth, and receiving low returns on their savings.[2] In this respect, our model provides a theory of savings that is consistent with four relevant facts that characterized China's development – most of which are direct consequences of the reforms enacted in the last forty years.

First, saving rates increased while fertility sharply declined (*Fact 1*). China's fertility rate decreased from 4.9 in 1975 to 1.7 in 2007, while life expectancy increased by ten years in the same period (Litao and Sixin, 2009). A major trigger of this impressive acceleration in population ageing was the one-child policy implemented since 1978, which changed family composition and reduced the number of births.

Second, chinese workers face an increased need to provide for old age with their own resources (*Fact 2*). A prominent cause is the reform of the industry sector implemented since the late 1980s, which gradually dismantled state owned entreprises and deleted cradle-to-grave social benefits for a huge fraction of workers (Ma and Yi, 2010).[3] Nowadays, state owned enterprises

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[2]The high savings rate reported in graph (a) of Figure 1 reflects the sum of high corporate savings and high household savings. Song et al. (2011) provide a theoretical explanation for high *corporate* savings based on the existence of capital market imperfections that generate high shares of firms' retained profits. Our claim on the lack of theories refers, instead, to the analysis of *household* savings, which is the focus of our model. At present, household savings is the single largest component of total savings and according to Yang (2012), the increase in the rate of household savings from 2000 to 2008 is the most important contribution to the overall increase in the Chinese savings rate in the same period.

[3]The reform implied massive layoffs: the share of workers in state-owned entreprises was halved from 1995 to 2005, and the enterprise based cradle-to-grave social safety net shrank rapidly as a result (Ma and Yi, 2010). The pre-reform system is discussed, e.g., in James (2002: p. 56): "During the cultural revolution, the provision of old-age security (and other forms of social service) became a responsibility of each state enterprise [which] provided housing, medical care and old-age security to its workers. The same services where provided to its pensioners".

can no longer be relied on to provide medical care and old-age security. Meanwhile, private coverage is neither pervasive nor efficient: according to Oksanen (2010), less than 30% of all employees are covered by pension schemes, and systemic deficiencies put at risk even the security of those covered.

Third, a growing share of old-age care services is, and will increasingly need to be, purchased on the market (*Fact 3*). The share of health spending that households pay themselves increased from 16% in 1980 to 61% in 2001 (Blanchard and Giavazzi, 2006), and the growth in China's health spending is "one of the most rapid in world history" (Eggleston, 2012: p.4). The rising importance of private provision may itself be a side-effect of the one-child policy through changes in family composition.[4] But beyond its causes, the relevant consequence for our analysis is that the increased share of care services in private expenditures is driving structural change in production sectors. Graph (b) in Figure 1 shows that the labor share in health and social work relative to the labor share in manufacturing has doubled over 15 years: from 1993 to 2008 the share of workers in manufacturing decreased from 37% to 29%, whereas the employment share of health and social work increased from 2.8% to 4.7%. This type of sectoral change has generally been neglected as a possible determinant of China's saving rates whereas it plays an important role in our model's predictions.

Fourth, the income distribution is shifting in favor of young wage earners and in disfavor of the old (*Fact 4*). As observed by Li et al. (2012), the number of people in the labor force may have peaked already in 2011, and since 1998 wage growth has exceeded GDP growth. This implies a shift in the income distribution towards young workers (Song and Yang, 2010). Zhong (2011) associates the widening income differences between those working and those retired to the one-child policy and population aging, which has induced labor shortages.

Our model produces equilibrium dynamics that are fully consistent with Facts 1-4: capital accumulation in the manufacturing sector raises wages and shifts labor into the care sector, boosting saving rates via both higher income for young cohorts and higher expected future cost of care services. In particular, we study exogenous shocks that plausibly capture the effects of China's past reforms – namely, a reduction in the population growth rate, an increase in the minimum level of care to be purchased, or in the elasticity of utility to care services – and we show that these shocks induce higher capital per capita and that saving rates increase during the transition because capital accumulation is accelerated by the savings multiplier. These results suggest that the one-child policy and the dismantling of cradle-to-grave social benefits have fuelled China's saving rates in the last decades. By the same token, our theory predicts that the introduction of a new welfare system would imply reduced accumulation and then lower capital per capita in the long run. This last conclusion bears, however, an important caveat. The mechanism of the savings multiplier implies that dynamic inefficiency – i.e., overaccumulation

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[4]By changing family composition and reducing the birth rate, the one-child policy drastically reduced the scope for family provided care during a period in which the need for such care was rapidly increasing. More and more families now consist of four grandparents, two parents and one child, making the private provision of care a necessity.

in the long run – is more likely to arise in our model with respect to the canonical OLG model. Therefore, implementing a social security system is not *a priori* welfare-reducing in terms of private utility.

The value added of our analysis lies in the use of the general equilibrium framework. In our model, the economy's equilibrium path brings together Facts 1-4 and combines them with a precise causal order. The existing empirical literature, instead, provides very valuable information on each of these facts but typically focuses on one single mechanism in isolation from the others, thus failing to deliver a complete picture. Kraay (2000) documents the link between the increased need to provide for old age and the dismantling of state-owned entreprises; Modigliani and Cao (2004) find a strong effect of the one-child policy on the needs to save for retirement; Choukhmane et al. (2013) establish a causal effect from lower fertility to higher savings; Blanchard and Giavazzi (2006) and Chamon and Prasad (2010) explain increased saving rates with the rising burden of defensive expenditures such as health care, education and housing; Song and Yang (2010) argue that the main reason for the rising saving rate is the shift in the income distribution in favor of young workers. Our paper is different, but complementary, to this line of research: none of the above mentioned contributions develops a general equilibrium model where capital accumulation affects subsequent saving rates, or note any of the two mechanisms behind the savings multiplier.

Our paper also relates to the debate on "communist capital accumulation". According to Acemoglu and Robinson (2012), growth in China has important similarities with growth in the former Soviet Union, based on high savings and massive capital accumulation, but being unsustainable if institutions are not reformed to be more inclusive. In fact, current investments rates in China are similar to those of the Soviet Union in the past, both exceeding 40% of GDP. In the Soviet Union forced savings through the suppression and collectivization of agriculture was key to fuel the capital accumulation. In China, on the other hand, agriculture has been decollectivized after 1978. Our paper points out how the combination of one-child policy with the dismantling of state enterprises without replacing them with a welfare system, may be an alternative channel to mobilize the savings required for "communist capital accumulation". To the best of our knowledge, this mechanism of forced savings has not previously been noted in the literature.

The rest of the paper is organized as follows. In Section 2 we set up the model. We show the static equilibrium of the model in Section 3. In Section 4 we study transitional capital accumulation and growth, and in Section 5 we define the savings multiplier and show how it magnifies the impact of exogenous shocks on long-run capital and income per capita. Section 6 investigates the introduction of a welfare state, dynamic inefficiency, and an extension to include endogenous growth. Section 7 concludes.[5]

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[5] **Detailed derivations and long proofs are collected in a separate Appendix for reviewers.**

2   The Model

In this section, we develop our model of savings and growth based on an overlapping-generations (OLG) structure that takes into account that when old, agents are in different needs from when young. In particular, when old agents need care. Thus, differently from the standard OLG framework pioneered by Diamond (1965) – henceforth termed the *canonical one-good model* – we separate between the production of goods and the production of care. One set of firms produces a *generic good*, used for investment and consumption of both young and old agents. The second set of firms provides *old-age care*.

2.1   Households

We consider an overlapping-generations environment where each agent lives two periods $(t, t + 1)$. Total population, denoted N_t, consists of N_t^y young and N_t^o old agents, and grows at the exogenous net rate $n > -1$;

$$N_t = N_t^y + N_t^o, \quad N_t^y = N_t^o \left(1 + n\right), \quad N_{t+1} = N_t \left(1 + n\right). \tag{1}$$

Households purchase two types of goods over their life-cycle: a generic consumption good and old-age care services. The generic good is consumed in both periods of life. Old-age care services, instead, are exclusively purchased by old agents. The utility of an agent born at the beginning of period t takes the additive form

$$U_t \equiv u\left(c_t\right) + \beta v\left(d_{t+1}, h_{t+1} - \bar{h}\right), \tag{2}$$

where c_t and d_{t+1} represent consumption levels of the generic good in the first and second period of life, respectively, h_{t+1} is the amount of old-age care consumed when old, $\bar{h} \geqslant 0$ is the *minimum requirement* – i.e., the minimum amount of old-age care required by old agents – and $\beta \in (0, 1)$ is the private discount factor between young and old age. A constraint of the consumer problem is that the minimum requirement is at least weakly satisfied,

$$h_{t+1} - \bar{h} \geqslant 0. \tag{3}$$

As is standard, we first study existence and uniqueness of interior equilibria where old-age care obeys (3). We then verify ex-post the conditions under which $h_{t+1} > \bar{h}$ holds.[6] The case where $\bar{h} = 0$, so that there is no minimum old-age care requirement, is of special interest. As we will see, this case transparently isolates what we term the intergenerational distribution effect in our model. For this reason, when we study the dynamics of the model in Section 4, we first put

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[6]In fact, in our main model which is the neoclassical case with constant returns to scale in generic-good production, there always exists a stable long-run equilibrium in which the allocation of labor between generic-good and health-care production exhibits stable shares consistent with the interior solution $h_{t+1} > \bar{h}$. We discuss cases where this may not be the case in Section 6.3, where we extend the model to allow for linear returns to capital at the aggregate level. Then, under certain conditions, the accumulation process may drive the economy towards long-run equilibria where labor is pushed away from the health-care sector so that the constraint $h_{t+1} - \bar{h} \geqslant 0$ becomes binding.

emphasis on this case, before we turn to the more general case of $\bar{h} \geqslant 0$, in which what we term the old-age requirement effect is also present.

We assume that only young agents work, supplying inelastically one unit of homogeneous labor. The only source of income in the second period of life is interest on previous savings. Personal lifetime income is entirely consumed at the end of the second period. Taking the consumption good as the numeraire in each period, the budget constraints read

$$c_t = w_t - s_t, \tag{4}$$

$$s_t R_{t+1} = d_{t+1} + p_{t+1} h_{t+1}, \tag{5}$$

where w_t is the wage rate, s_t is savings, R_{t+1} is the (gross) rate of return to saving, and p_{t+1} is the price of old-age care. Savings consist of physical capital, which as in the one-good OLG model is homogeneous with the generic consumption good. Assuming full depreciation within one period, market clearing requires that aggregate capital at the beginning of period $t+1$ equals aggregate savings of the young agents in the previous period, $K_{t+1} = N_t^y s_t$.

In order to make our new mechanisms as transparent as possible, we consider a specific, yet flexible form of preferences:

$$u(c_t) \equiv \log c_t, \tag{6}$$

$$v\left(d_{t+1}, h_{t+1} - \bar{h}\right) \equiv \log \left[\gamma \left(d_{t+1}\right)^{\frac{\sigma-1}{\sigma}} + (1-\gamma) \left(h_{t+1} - \bar{h}\right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \tag{7}$$

where $\gamma \in [0,1]$ is a weighting parameter and $\sigma > 0$ is the elasticity of substitution between consumption goods and care services in the second period of life: d_{t+1} and h_{t+1} are strict complements if $\sigma < 1$, strict substitutes if $\sigma > 1$. In the limiting case $\sigma \to 1$, the term in square brackets reduces to the Cobb-Douglas form $(d_{t+1})^\gamma (h_{t+1})^{1-\gamma}$. When $\bar{h} > 0$, the income elasticity of old-age care falls short of unity, resembling the case in Acemoglu et al. (2013), who estimate the income elasticity of health spending to 0.7.

Assumptions (6)-(7) imply two fundamental properties. First, we can treat the canonical one-good model as a special case: letting $\gamma = 1$ (and $\bar{h} = 0$), old-age care services disappear from private utility and, hence, are not produced in equilibrium. Second, the utility functions (6)-(7) exhibit a unit elasticity of intertemporal substitution. This property allows us to describe the effects of old-age care on saving rates in the clearest way. Setting $\gamma = 1$, we obtain the logarithmic version of the canonical model, in which the saving rate is constant over time because consumption propensities are independent of the interest rate.[7] Hence, in the general case $0 < \gamma < 1$, any departure from constant saving rates in the model is exclusively due to the inclusion of old-age care services.

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[7] More precisely, the savings rate of the young is constant with logarithmic preferences. When production is Cobb-Douglas, the income share of the young is constant, and thus also the aggregate savings rate is constant.

2.2 Production Sectors

Old-age care is labor intensive. In our framework this implies that the factor price of interest to old agents is not only the interest rate, but also the wage rate. This contrasts with standard one-good OLG models. There, old agents are on the supply side of the capital market, and the only relevant factor price when old is the (real) interest rate. In the present model, old agents are still on the supply side of the capital market, but since when old they need care, they are in addition on the demand side of the labor market. This implies that the wage rate is also important for old agents. To clarify this, and to capture in a simple way that care is more labor intensive than the production of the generic consumption and investment good, we assume that care services are produced with labor as the only factor of production.[8]

We denote by ℓ_t the fraction of workers employed in the generic sector, and by $1 - \ell_t$ the fraction employed in the care sector. Perfect labor mobility and perfectly competitive conditions in the labor market ensure wage equalization in equilibrium. In the old-age care sector, there is a simple constant returns to scale production technology:

$$H_t \equiv \eta \left(1 - \ell_t\right) N_t^y, \tag{8}$$

where H_t is the aggregate output of care services, and $\eta > 0$ is a constant labor productivity parameter.

In the generic good sector, we consider a specification displaying constant returns to scale at the firm level. A continuum of firms, indexed by $j \in [0, J]$, exploits the same Cobb-Douglas technology

$$X_t^j \equiv (k_t^j)^\alpha (a_t \ell_t^j N_t^y)^{1-\alpha} \text{ for each } j \in [0, J], \tag{9}$$

where X_t^j is the output of the generic good produced by the j-th firm, k_t^j and $\ell_t^j N_t^y$ are the amounts of physical capital and labor employed at the firm level, $\alpha \in (0, 1)$ is an elasticity parameter, and a_t is labor productivity in the generic-good sector.

2.3 Labor Productivity

Specification (9) assumes that the generic good technology displays constant returns to scale *at the firm level*, so that income shares are determined according to standard zero-profit conditions. In the main model of our paper we also make the standard neoclassical assumption of constant returns to scale at the *aggregate level*. We then impose that a_t equals an exogenous constant $B^{\frac{1}{1-\alpha}}$ in each period: the generic production sector exhibits strictly diminishing marginal returns to capital also at the aggregate level, and aggregate sectoral output $X_t \equiv J X_t^j$ is given by

$$X_t = B \left(K_t\right)^\alpha \left(\ell_t N_t^y\right)^{1-\alpha} \tag{10}$$

where $K_t \equiv J k_t^j$ is aggregate capital and $\ell_t \equiv J \ell_t^j$ is aggregate labor employed in the generic sector. This is the setup of the canonical model since the seminal work of Diamond (1965).

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[8]For a two-sector OLG model with capital in both sectors, as well as the existence and stability properties of such models, see Galor (1992).

In Subsection 6.3 we extend the model to allow for endogenous growth in its simplest fashion. Following Romer (1989), we include learning-by-doing whereby the productivity of workers employed in the generic sector increases with the amount of capital that each of these workers uses. In this case the labor productivity is governed by the spillover function $a_t = A^{\frac{1}{1-\alpha}} K_t / (\ell_t N_t^y)$, where A is an exogenous constant. Since a_t is taken as given at the firm level, income shares are still determined by the usual zero-profit conditions, but aggregate sectoral output is proportional to aggregate capital:

$$X_t = AK_t. \tag{11}$$

We next describe the equilibrium conditions that hold independently of the assumed technology for the generic good, then put our main emphasis on the neoclassical case, before returning to the extension of the model to endogenous growth in Subsection 6.3.

3 Static Equilibrium

This section discusses the static equilibrium conditions holding in each period for a given stock of capital per worker. We first study the profit-maximizing conditions for firms, the utility-maximizing conditions for households, the labor market equilibrium, and the goods market equilibrium. We then study the joint (static) equilibrium of all the markets, the implications for the aggregate savings rate, and finally the implied mapping to capital accumulation.[9]

3.1 Firms

In the service sector for old-age care, the technology (8) implies a wage that equals the market price of services times the labor productivity,

$$w_t = p_t \eta. \tag{12}$$

Market clearing requires that total output of old-age care services matches aggregate demand by old agents, $H_t = N_t^o h_t$. The existence of a minimum requirement, $h_t \geqslant \bar{h}$, requires that total production H_t exceeds $N_t^o \bar{h}$, which implies a constraint on sectoral employment shares: using the production function (8), we obtain

$$\ell_t \leqslant \frac{\eta (1+n) - \bar{h}}{\eta (1+n)} \equiv \ell^{\max}, \tag{13}$$

where $\ell^{\max}$ is the maximum level of employment in the generic sector that is compatible with a level of old-age care output equal to the minimum requirement.[10] In the remainder of the analysis, we will work under the parameter restriction

$$\bar{h} \leqslant \eta (1+n), \tag{14}$$

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[9]Unless otherwise specified, all equations in this section are valid in the neoclassical case as well as in the AK-case. Thus, to avoid repetitions when we extend the model to endogenous growth in Subsection 6.3, in the present section we continue to use a_t for the labor productivity, without specifying if growth is neoclassical or endogenous (when not necessary).

[10]Formally, the level of health-care output equal to the minimum requirement is $H_t^{\min} \equiv \eta (1 - \ell^{\max}) N_t^y = N_t^o \bar{h}$.

which implies $\ell^{\max} \geqslant 0$. By construction, when the minimum requirement is $\bar{h} = 0$, we have $\ell^{\max} = 1$.

In the generic good sector each firm maximizes own profits $X_t^j - R_t k_t^j - w_t \ell_t^j N_t^y$ subject to technology (9). Denoting capital *per young* agent by $\kappa_t \equiv K_t / N_t^y$ and , respectively, the zero-profit conditions in the sector can be aggregated across firms and written as

$$w_t = a_t^{1-\alpha} (1-\alpha) (\kappa_t/\ell_t)^\alpha = (1-\alpha) (x_t/\ell_t), \qquad (15)$$

$$R_t = a_t^{1-\alpha} \alpha (\ell_t/\kappa_t)^{1-\alpha} = \alpha (x_t/\kappa_t), \qquad (16)$$

where $x_t \equiv X_t / N_t^y$ is sectoral output *per young* agent. Aggregating the incomes of both sectors, we thus have

$$\frac{Y_t}{N_t^y} = w_t + R_t \kappa_t = x_t \left(\frac{1-\alpha}{\ell_t} + \alpha \right), \qquad (17)$$

where Y_t is aggregate income, which coincides with the total value of goods and services produced in the economy.[11]

3.2 Consumers

Each agent maximizes (2) subject to the budget constraints (4)-(5). Denoting the derivative of the u-function with respect to c_t by u_{c_t}, and so on, the solution to this problem yields two familiar first order conditions; the Keynes-Ramsey rule, $u_{c_t} = \beta R_{t+1} v_{d_{t+1}}$, and an efficiency condition establishing the equality between the price of care services and the marginal rate of substitution with second-period generic goods consumption, $v_{h_{t+1}} / v_{d_{t+1}} = p_{t+1}$. Under preferences (6)-(7), we show in the Appendix that these conditions result in the following relationships.

Consumption and savings of young agents are given by

$$c_t = \frac{1}{1+\beta} \left(w_t - \frac{p_{t+1}}{R_{t+1}} \bar{h} \right) \text{ and } s_t = \frac{1}{1+\beta} \left(\beta w_t + \frac{p_{t+1}}{R_{t+1}} \bar{h} \right). \qquad (18)$$

Note that when $\bar{h} = 0$, these expressions are equivalent to those in the simplest version of the canonical OLG model, where young agents save a constant fraction of their wage income, which is then used to provide old age consumption.[12] When $\bar{h} > 0$, individual decisions on c_t and s_t are no longer fixed proportions of young age income. Young age consumption is lower, and savings higher, the larger is $\bar{h}$. More interesting, the strength of the effect is related to the *future* relative factor price, since $p_{t+1}/R_{t+1} = \eta w_{t+1}/R_{t+1}$. A high future wage w_{t+1}, and low returns on savings R_{t+1}, imply that much must be saved today in order to purchase the minimum amount of care tomorrow. We term this the *old-age requirement effect*. The old-age

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[11] Defining the value of total output as $Y_t \equiv X_t + p_t H_t$, zero profits in both sectors implies $Y_t = w_t N_t^y + R_t K_t$ and therefore expression (17).

[12] As we will return to, however, this does not imply that the dynamics are equivalent to the canonical OLG model. As we will see, these are quite different also in the case where $\bar{h} = 0$, because in our model the *aggregate* savings rate is not constant due to our intergenerational income distribution effect.

requirement effect implies that future relative factor prices affect present savings.[13]

Turning next to generic consumption in the second period of life, each old agent purchases

$$d_t = (1 + n) \left[\ell_t - (1 - \alpha) \right] a_t^{1-\alpha} \left(\kappa_t / \ell_t \right)^{\alpha}, \tag{19}$$

which is the residual (per-old) output of the generic sector after consumption and savings of young agents have been subtracted. Result (19) implies that second-period consumption is positive only if $\ell_t > 1 - \alpha$, which, as we will see, always turns out to be the case in equilibrium.

Finally, the relative demand for old-age care links the old agents' expenditure shares over the two goods to their relative price:

$$\frac{p_t \cdot \left(h_t - \bar{h} \right)}{d_t} = \left(\frac{1 - \gamma}{\gamma} \right)^{\sigma} p_t^{1-\sigma}. \tag{20}$$

Expression (20) shows that the expenditure share of old agents on net health care, $h_t - \bar{h}$, increases (decreases) with the price when the two goods are complements (substitutes). The reason is that a ceteris paribus increase in p_t always reduces the 'physical consumption ratio' between net care and generic consumption, $\left(h_t - \bar{h} \right) / d_t$, but in the usual fashion the final effect on the 'expenditure ratio' $p_t \left(h_t - \bar{h} \right) / d_t$ depends on the elasticity of the relative demand for net care. Under complementarity, the demand is relatively rigid: if p_t increases, the price effect dominates the quantity effect and the expenditure share of net care increases. Under substitutability, instead, net old-age care demand is relatively elastic and the quantity effect dominates: an increase in p_t decreases the expenditure share of care. These substitution effects will imply that variations in the price of care have an impact on the labor allocation between the two production sectors.[14]

3.3 Labor Market

The labor demand schedules of both production sectors determine a unique equilibrium in the labor market. Combining (12) with (15), we obtain

$$p_t = (1/\eta) (1 - \alpha) a_t^{1-\alpha} \left(\kappa_t / \ell_t \right)^{\alpha} \equiv \Phi \left(\ell_t, \kappa_t; a_t \right). \tag{21}$$

Condition (21) establishes that, in equilibrium, the wage rate must be equalized between the two production sectors. In particular, (21) defines p_t as the level of the price of care ensuring

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[13] In particular, the feature that the future wage is relevant for individual consumption and savings decisions is in contrast to one-good versions of the OLG model, where the only future factor price relevant is the return to savings. Moreover, note that in general this feature is the result of old-age care in the model, and does not require $\bar{h} > 0$. For instance, with an intertemporal elasticity of substitution that falls short of one, a higher future wage would imply higher young age savings also in the case where $\bar{h} = 0$.

Also, to preview some intuition, note that since the future wage affects young age savings, it is already clear at this stage that the general equilibrium dynamics will be quite different from one-good OLG models. For instance, higher future wages implies higher savings and thus higher future capital stock, in turn increasing future wages even more.

[14] As usual substitution effects only disappear with Cobb-Douglas preferences: when $\sigma = 1$, the expenditure shares of generic goods and old-age care are independent of the relative price, and are exclusively determined by the relevant preference parameter γ.

equal wages between the two sectors for given levels of sectoral employment, capital per worker, and productivity.

The labor market equilibrium differs between the neoclassical case in our main model, and the extension to the AK case in Subsection 6.3. By substituting for the relevant value of labor productivity $a_t = B^{\frac{1}{1-\alpha}}$ the expression for the labor market equilibrium in the neoclassical case is given by

$$\Phi\left(\ell_t, \kappa_t\right) = \left(B/\eta\right)\left(1-\alpha\right)\left(\kappa_t/\ell_t\right)^{\alpha}, \tag{22}$$

while in the AK case this expression has to be replaced by

$$\Phi\left(\ell_t, \kappa_t\right) = \left(A/\eta\right)\left(1-\alpha\right)\left(\kappa_t/\ell_t\right). \tag{23}$$

In both cases the function $p_t = \Phi\left(\ell_t, \kappa_t\right)$ is strictly decreasing in ℓ_t; for a given capital per young κ_t, higher employment in the generic sector decreases the marginal productivity of labor, implying a lower wage, and thus a lower price of care.

3.4 Goods Markets

In the Appendix we show that solving the demand relationship (20) for the price of care, and substituting $p_t h_t / d_t$ with the market-clearing and zero-profit conditions holding for the producing firms, we obtain

$$p_t = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\sigma}{\sigma-1}}\left[\frac{\left(1-\alpha\right)\left(\ell^{\max}-\ell_t\right)}{\ell_t - \left(1-\alpha\right)}\right]^{\frac{1}{1-\sigma}} \equiv \Psi\left(\ell_t\right). \tag{24}$$

This expression defines p_t as the price of care that ensures equilibrium in the goods market.[15] The most important insight of (24) is that the function $p_t = \Psi\left(\ell_t\right)$ is strictly decreasing when $\sigma < 1$, and strictly increasing when $\sigma > 1$. When $\sigma < 1$ the price of care is positively related to the employment share in the care sector $1 - \ell_t$. The reason is that a ceteris paribus increase in p_t increases the expenditure share old consumers devote to care services relative to generic consumption and, consequently, attracts labor in the care sector. When $\sigma > 1$, in contrast, a higher price of care means a lower expenditure share of care, and thus less labor in the care sector and more labor in the generic sector.[16]

3.5 Employment and Capital Co-Movements

Consider now the joint equilibrium of the markets for labor and for goods. The two relevant conditions, (22) and (24) in the neoclassical case, imply that the price of health care and the employment shares of the two sectors in each period t depend on the level of capital per worker

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[15]Note that the term in square brackets only contains ℓ_t because, with Cobb-Douglas technologies, the sector allocation of labor alone determines the output ratio $X_t / p_t H_t$. If we deviate from Cobb-Douglas technologies, the term in square brackets would also contain capital employed in generic production: see the derivation of (24) in the Appendix.

[16]It should be noted that, in the special case of unit elasticity of substitution, $\sigma = 1$, expression (24) does not hold because price and quantity effects on the demand side balance each other. As a result, the equilibrium between demand and supply in the goods market is characterized by constant employment shares, with $\ell_t = \frac{(1-\alpha)(\gamma\ell^{\max}+1-\gamma)}{\gamma(1-\alpha)+1-\gamma}$ at each t.

κ_t. Formally, the employment share of the generic sector for a given level of κ_t, denoted by $\ell_t = \ell(\kappa_t)$, is the fixed point

$$\ell(\kappa_t) \equiv \arg\text{solve}_{\{\ell_t \in (1-\alpha, \ell^{\max})\}} [\Phi(\ell_t, \kappa_t) = \Psi(\ell_t)]. \tag{25}$$

Our assumptions guarantee the existence and uniqueness of this fixed point – a result that is shown in the Appendix and that can be verified in graphical terms in Figure 2. On the one hand, the function $\Phi(\ell_t, \kappa_t)$ is strictly decreasing in ℓ_t and exhibits positive vertical intercepts at the boundaries of the relevant interval $\ell_t \in (1-\alpha, \ell^{\max})$. On the other hand, the function $\Psi(\ell_t)$ is decreasing (increasing) under complementarity (substitutability) with limits

$$\lim_{\ell_t \to 1-\alpha} \Psi(\ell_t) = \quad \{ \quad \infty \quad \text{if } \sigma < 1; \quad 0 \quad \text{if } \sigma > 1 \quad \},$$

$$\lim_{\ell_t \to \ell^{\max}} \Psi(\ell_t) = \quad \{ \quad 0 \quad \text{if } \sigma < 1; \quad \infty \quad \text{if } \sigma > 1 \quad \},$$

These properties[17] ensure the existence and uniqueness of the fixed point $\Psi(\ell_t) = \Phi(\ell_t, \kappa_t)$, and that it is contained in the relevant interval $\ell \in (1-\alpha, \ell^{\max})$. The fixed point (25) simultaneously determines employment shares and the price of care, which is measured along the vertical axis of Figure 2. Substituting $\ell(\kappa_t)$ in $\Psi(\ell_t)$ or in $\Phi(\ell_t, \kappa_t)$ we obtain the equilibrium price of care for given capital per worker,

$$p(\kappa_t) \equiv \Psi(\ell(\kappa_t)) = \Phi(\ell(\kappa_t), \kappa_t). \tag{26}$$

Even though we have not yet specified whether and how capital grows, result (26) clarifies how capital accumulation affects the price of care and employment shares:

Proposition 1 *An equilibrium trajectory with positive accumulation implies a rising price of care. Under complementarity the employment share in the generic sector is decreasing. Under substitutability the employment share in the generic sector is increasing;*

$$\kappa_{t+1} > \kappa_t \qquad \Longleftrightarrow \qquad p_{t+1} > p_t$$

and

$$\kappa_{t+1} > \kappa_t \Rightarrow \left\{ \begin{array}{l} \ell_{t+1} < \ell_t \ \text{if } \sigma < 1 \\ \ell_{t+1} > \ell_t \ \text{if } \sigma > 1 \end{array} \right\}$$

Proof. The proposition is proved in graphical terms by means of a comparative-statics exercise.[18] Because $\Phi(\ell, \kappa)$ is positively related to κ, a higher stock of capital per young implies an up-rightward shift in the $\Phi(\ell, \kappa)$ curves in Figure 2. The new equilibrium price $p(\kappa)$ is higher in all cases, but sectoral employment shares react differently depending on the value of σ. The employment share of the generic sector $\ell(\kappa)$ increases under complementarity, decreases under substitutability:

$$\ell'_\kappa \equiv \frac{\mathrm{d}\ell(\kappa_t)}{\mathrm{d}\kappa_t} \quad < 0 \quad \text{if } \sigma < 1; \quad > 0 \quad \text{if } \sigma > 1 \ .$$
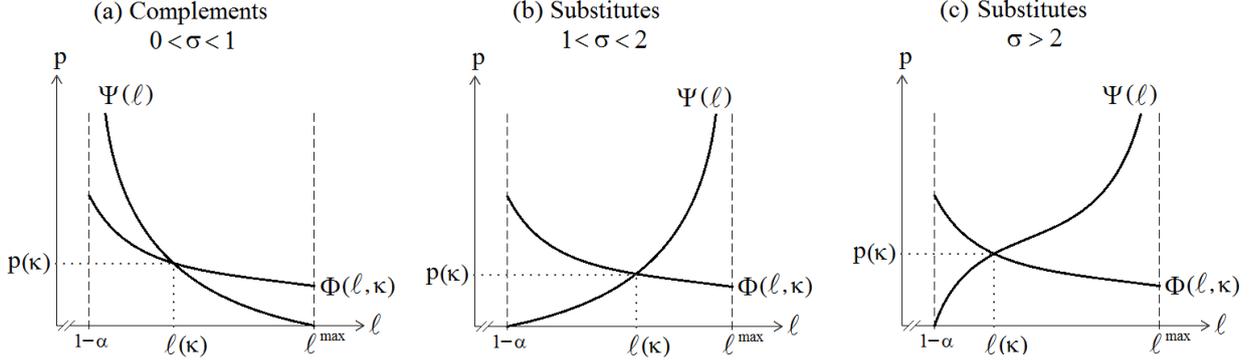
Figure 2: Static equilibrium: determination of ℓ_t and p_t for given κ_t. Qualitatively, the graphs do not change between the neoclassical and the AK model. The case of strong substitution ($\sigma > 2$) implies local concavity of $\Psi(\ell)$ for low ℓ without altering existence, uniqueness, and comparative-statics properties.

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The intuition is that an increase in capital per young expands the production frontier of the generic good, and thereby increases the price of care. Under complementarity, old agents react to the price increase by raising the share of expenditure on net old-age care, which decreases the employment share in the generic sector $\ell(\kappa)$. Under substitutability, instead, old agents reduce the expenditure share on net care, and employment in the generic sector therefore grows. It is easily verified that the direction of these capital and employment co-movements is fully reversed when we consider an equilibrium trajectory with decumulation of capital per young – that is, when $\kappa_t < \kappa_{t-1}$.[19]

3.6 Static Equilibrium Comparative Statics

For a given capital stock, the static equilibrium labor allocation depends on the parameters in the model. In particular, for later use we investigate how it depends on productivity B, on population growth n, and on the level of the minimum requirement $\bar{h}$. The properties of $\ell(\kappa_t) = \ell(\kappa_t; B, n, \bar{h})$ are summarized in the following Proposition:[20]

Proposition 2 *In the static equilibrium with given κ_t,*

$$\frac{d\ell(\kappa_t; B, n, \bar{h})}{dB} \equiv \ell'_B \; < 0 \quad if \; \sigma < 1; \quad > 0 \quad if \; \sigma > 1 \; , \tag{27}$$

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[17] Along with the further concavity properties of both curves described in the Appendix.

[18] Proposition 1 is equivalently proved by differentiating the equilibrium condition $\Psi(\ell(\kappa_t)) = \Phi(\ell(\kappa_t), \kappa_t)$. The exact relationship between κ and ℓ is reported in expression (36) below, and indeed implies that $\ell'_\kappa \equiv d\ell(\kappa_t)/d\kappa_t$ is strictly negative (positive) under complementarity (substitutability).

[19] Note that all the properties in this subsection, and therefore the identical results established in Proposition 1 as well as the proof, also hold in the AK case in our model: the co-movements of employment shares, price of health care and capital per worker are the same in both variants of the model.

[20] Again, this proposition is also valid if the productivity term B from the neoclassical version of the model is replaced by the productivity term A in the AK version of the model.

$$\frac{d\ell\left(\kappa_t; B, n, \bar{h}\right)}{d\bar{h}} \equiv \ell'_{\bar{h}} < 0 \tag{28}$$

and

$$\frac{d\ell\left(\kappa_t; B, n, \bar{h}\right)}{dn} \equiv \ell'_n > 0 \ \text{if} \ \bar{h} > 0 \quad (= 0 \ \text{if} \ \bar{h} = 0). \tag{29}$$

Proof. Also this proposition can be proved in graphical terms. An increase in B implies an upward shift in $\Phi\left(\ell, \kappa\right)$ in Figure 2. The employment share, ℓ, increases when $\sigma < 1$ while it decreases when $\sigma > 1$. Changes in population growth, n, and minimum care requirement, $\bar{h}$, operate through $\ell^{\max}$ that appears in the expression for $\Psi\left(\ell\right)$ in equation (24). An increase in $\ell^{\max}$ shifts $\Psi\left(\ell\right)$ to the right, increasing ℓ. As $\ell^{\max} \equiv 1 - \frac{\bar{h}}{\eta(1+n)}$, $\ell^{\max}$ increases with a lower $\bar{h}$ or with a higher n (provided that $\bar{h} > 0$). ■

A higher productivity B expands production possibilities of generic goods. When $\sigma < 1$, labor is pushed out of the generic sector, as consumers want to utilize the increased production possibilities to consume more services from the care sector. When $\sigma > 1$ in contrast, labor is drawn into the generic sector, since in this case old agents prefer less care but more generic goods.

The intuition for the effects working via $\ell^{\max}$ are intuitive. When a larger fraction of workers are needed in order to satisfy the minimum service requirement, the care sector will employ more workers.

3.7 Saving Rates and Accumulation

Before studying in detail the dynamics, it is instructive to describe the general relationships between saving rates, capital accumulation and sectoral employment shares. Considering the economy's aggregate income (17) and the wage rate (15), the total labor share accruing to young agents is given by

$$\frac{w_t N_t^y}{Y_t} = \frac{(1-\alpha)\frac{x_t}{\ell_t}}{x_t \left(\frac{1-\alpha}{\ell_t} + \alpha\right)} = \frac{1-\alpha}{1 - \alpha\left(1 - \ell_t\right)}, \tag{30}$$

Equation (30) shows that, in static equilibrium, an increase in the generic sector employment share ℓ_t reduces the total income share of young agents. The intuition is that if labor moves from the care sector to generic production, the return to capital increases relative to the wage rate. There is, thus, a shift in the income distribution away from the young towards the old. We term this effect the *intergenerational distribution effect*.

Since it is the young who save, the intergenerational distribution effect directly influences the economy's saving rate (and will, as we shall see, have important implications for capital accumulation). The savings rate, termed θ_t and defined as aggregate savings relative to the total value of production, is found by using the saving function in (18) and expression (30), and then inserting for $\ell^{\max}$ from (13):

$$\theta_t \equiv \frac{N_t^y s_t}{Y_t} = \underbrace{\frac{\beta\left(1-\alpha\right)}{1+\beta}}_{\text{Canonical model}} \cdot \underbrace{\frac{1}{1 - \alpha \cdot \left(1 - \ell_t\right)}}_{\text{Intergenerational Distribution}} \cdot \underbrace{\Gamma\left(\frac{\bar{h}}{\ell_{t+1}}\right)}_{\text{Old-age Requirement}}, \tag{31}$$

where

$$\Gamma\left(\frac{\bar{h}}{\ell_{t+1}}\right) \equiv \left[1 - \frac{(1-\alpha)}{\alpha\,(1+\beta)\,\eta(1+n)}\frac{\bar{h}}{\ell_{t+1}}\right]^{-1}, \quad \Gamma' > 0, \quad \Gamma(0) = 1. \tag{32}$$

Expression (31) is a semi-reduced form showing that the savings rate is negatively related to both ℓ_t and ℓ_{t+1}. The function Γ captures the savings induced if there is a minimum health requirement. When $\bar{h} = 0$, Γ reduces to unity. The derivative is positive, and thus when $\bar{h} > 0$, then $\Gamma > 1$.

To explain the intuition it is instructive to compare the result in (31) to the savings rate in the canonical OLG model with logarithmic preferences and Cobb-Douglas technology. There, the young save a fraction $\beta/(1+\beta)$ of their income, and the income share of the young is $1 - \alpha$. The savings rate is therefore, in this case, given by the first of the three terms on the right hand side of (31), and it is time independent.

The present model implies that the savings rate is, in general, not constant over time. Moreover, it is always higher than in the canonical model for two reasons; the intergenerational distribution effect and the old-age requirement effect. First, as seen by the second term on the right hand side of (31), the presence of employment in the care sector implies higher labor demand, shifting the income distribution in favor of the young, and thus increasing savings. Second, as seen by the third term on the right hand side of (31), with $\bar{h} > 0$, as we have seen from (18), the young have an additional savings motive in that they need some minimum amount of old-age care, increasing the savings rate further.[21] The old-age requirement effect on savings is stronger the lower is ℓ_{t+1}, because lower future employment in the generic sector implies higher future wages, increasing the cost of purchasing the minimum requirement of care. The expected increase in the cost of health care in period $t + 1$ prompts young agents to save more in period t and, therefore, to accumulate more capital.

The natural question concerns the general-equilibrium impact of both these mechanisms on economic growth. In this respect, the market-clearing condition equating investment to savings implies that capital per worker obeys the dynamic law

$$\kappa_{t+1} = \frac{\theta_t Y_t}{1 + n}, \tag{33}$$

Next periods capital per young is determined by this periods savings adjusted for population growth.

The next section discusses capital accumulation in the neoclassical variant of the model, while Subsection 6.3 extends the dynamics to the AK case.

4 Neoclassical Growth

In the neoclassical case labor productivity in the generic sector equals $a_t = B^{\frac{1}{1-\alpha}}$ in each period. In this framework, when the economy reaches a long-run equilibrium where capital per worker

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[21] In the Appendix we show that restriction (14) and $\ell_{t+1} > 1 - \alpha$ implies that $(1-\alpha)\bar{h} < \alpha\,(1+\beta)\,\eta(1+n)\ell_{t+1}$.

is constant, generic production grows at the exogenous rate of population growth. Subsections 4.1-4.3 derive the stability properties of the long-run steady state and show that, given an initial stock below the steady-state level, capital per worker grows monotonously. We also show that under complementarity, these transitional dynamics are characterized by increasing savings rates. Under substitutability, on the other hand, savings rates decrease during the transition to steady-state. The intergenerational distribution effect and the old-age requirement effect both contribute to these results.

Compared to the canonical OLG model the dynamics are more involved: since increased capital increases savings rates and thereby capital further, this opens for the possibility of (local) instability and multiple steady states. We show, however, that a departure from uniqueness and stability of the steady state can only occur under unreasonable high values of the elasticity of capital in generic production α.[22]

4.1 Accumulation Law

The equilibrium path of capital is determined by the saving decisions of young agents. Inserting from (31) and (17) in (33), we obtain a semi-reduced form of the accumulation law of capital per worker, which links κ_{t+1} to the previous stock κ_t and to the sectoral employment levels in the two periods:

$$\kappa_{t+1} = \underbrace{\frac{B\beta(1-\alpha)}{(1+\beta)(1+n)}\kappa_t^{\alpha}}_{\text{Canonical model}} \cdot \underbrace{\ell_t^{-\alpha}}_{\text{Intergen. Distr.}} \cdot \underbrace{\Gamma\left(\frac{\bar{h}}{\ell_{t+1}}\right)}_{\text{Old-age requirement effect}}. \tag{34}$$

This expression decomposes the accumulation law of capital in three parts. The first term on the right hand side of (34) is the dynamic law in the canonical one-good model: if we eliminate the care sector by setting $\ell_t = 1$ and $\bar{h} = 0$, capital per worker evolves according to this stable monotonic relationship, and the saving-output ratio is constant by virtue of constant income share of the young and logarithmic intertemporal preferences.

The second and third terms on the right hand side of (34) again directly follow from the intergenerational distribution effect and the old-age requirement effect. An increase in ℓ_t reduces κ_{t+1} because a lower current wage reduces young agents' income, and thereby, current savings. An increase in ℓ_{t+1} reduces κ_{t+1} because a lower future wage reduces the expected future cost of health care, and thereby, current savings.

Recalling result (25), equilibrium employment shares are a function of the capital stock per worker in each period. Substituting $\ell_t = \ell(\kappa_t)$ and $\ell_{t+1} = \ell(\kappa_{t+1})$ into (34), we obtain the accumulation law

$$\kappa_{t+1} = \frac{B\beta(1-\alpha)}{(1+\beta)(1+n)}\kappa_t^{\alpha}[\ell(\kappa_t)]^{-\alpha}\Gamma\left(\frac{\bar{h}}{\ell(\kappa_{t+1})}\right). \tag{35}$$

Expression (35) implies that capital dynamics crucially depend on how sectoral employment

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[22] Nevertheless, for completeness we also solve the dynamics for this case in the Appendix.

shares react to variations in capital per worker. In this respect, the relevant elasticity is[23]

$$\frac{\ell'_\kappa\left(\kappa_t\right)\kappa_t}{\ell\left(\kappa_t\right)} = \frac{1}{1 - \frac{1}{1-\sigma}\frac{1}{\alpha}Q_1}\left\{ \begin{array}{l} < 0 \text{ if } \sigma < 1 \\ > 0 \text{ if } \sigma > 1 \end{array} \right\}, \tag{36}$$

where $Q_1 \equiv \frac{\ell_t}{\ell_t - (1-\alpha)} \cdot \frac{\ell^{\max} - (1-\alpha)}{\ell^{\max} - \ell_t} > 1$. The slope of the accumulation law can be found by taking the elasticity of (35) with respect to κ_t and κ_{t+1}, which yields[24]

$$\frac{\mathrm{d}\kappa_{t+1}}{\mathrm{d}\kappa_t}\frac{\kappa_t}{\kappa_{t+1}} = \frac{\alpha - \alpha\frac{\ell'_\kappa(\kappa_t)\kappa_t}{\ell(\kappa_t)}}{1 + \frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\kappa_{t+1})}\frac{\ell'_\kappa(\kappa_{t+1})\kappa_{t+1}}{\ell(\kappa_{t+1})}}. \tag{37}$$

Starting with the numerator, we see that the direct effect on κ_{t+1} of an increase in κ_t is larger under complementarity, i.e. when $\ell'_\kappa\left(\kappa_t\right) < 0$. When $\bar{h} > 0$, there is also an indirect effect via the increase in $\ell\left(\kappa_{t+1}\right)$, captured in the denominator.

To present the intuition in the most transparent way we first, in the next subsection, investigate the special case where $\bar{h} = 0$, and thus $\Gamma = 1$. This isolates the intergenerational distribution effect, and shows how this increases the steady state capital stock. In Subsection 4.3 we then expand the model to the case where $\bar{h} > 0$. This shows how the old-age requirement effect further increases the steady state capital stock.

4.2 Dynamics without Minimum Requirement

When there is no minimum health-care requirement for old agents, capital accumulation obeys a fairly simple dynamic law. In the main text, we assume that the elasticity of capital in generic production is not too high, that is:[25]

Assumption 1: $\alpha < \frac{3}{4}$.

This assumption is sufficient (but not necessary) for the steady state to be unique.[26] The next Proposition then establishes that the steady state is globally stable: under both complementarity and substitutability, the economy converges towards a long-run equilibrium in which capital per worker, the price of health care and employment shares are constant.

Proposition 3 *In the neoclassical case with $\bar{h} = 0$, capital per worker obeys*

$$\kappa_{t+1} = \frac{\beta\eta}{(1+n)(1+\beta)}p\left(\kappa_t\right), \tag{38}$$

where $p\left(\kappa_t\right)$ is the price of health care determined by (26). Under Assumption 1 the steady state $\kappa_{ss} = \frac{\beta\eta}{(1+n)(1+\beta)}p\left(\kappa_{ss}\right)$ is unique and globally stable, implying

$$\lim_{t\to\infty}\kappa_t = \kappa_{ss}, \quad \lim_{t\to\infty}\ell_t = \ell\left(\kappa_{ss}\right), \quad \lim_{t\to\infty}p_t = p\left(\kappa_{ss}\right).$$

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[23] Expression (36) is obtained by differentiating the equilibrium condition $\Psi\left(\ell\left(\kappa_t\right)\right) = \Phi\left(\ell\left(\kappa_t\right), \kappa_t\right)$ and is fully derived in the Appendix. The fact that $Q_1 > 1$ directly follows from the requirement $1 - \alpha < \ell_t < \ell^{\max}$ and it implies the signs reported in (36). Note that (36) yields an alternative proof of Proposition 1.

[24] Totally differentiating (35) yields $\frac{\mathrm{d}\kappa_{t+1}}{\kappa_{t+1}} = \alpha\frac{\mathrm{d}\kappa_t}{\kappa_t} - \alpha\frac{\partial\ell(\kappa_t)}{\partial\kappa_t}\frac{1}{\ell(\kappa_t)}\mathrm{d}\kappa_t - \frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell_{t+1}}\frac{\partial\ell(\kappa_{t+1})}{\partial\kappa_{t+1}}\frac{1}{\ell(\kappa_{t+1})}\mathrm{d}\kappa_{t+1}$, which can be rearranged to obtain (37).

[25] In the Appendix part B, we solve the general model for the case in which Assumption 1 is not satisfied. For more on stability properties in OLG models with one capital stock, see e.g. Galor and Ryder (1989).

[26] Under substitutability the steady state is always unique and stable.

During the transition, given a positive initial stock $\kappa_0 < \kappa_{ss}$, both capital per worker and the price of health care increase, whereas employment in the generic sector declines (increases) and the saving rate increases (declines) under complementarity (substitutability):

$$\kappa_{t+1} > \kappa_t, \quad p_{t+1} > p_t, \quad \left\{ \begin{array}{l} \ell_{t+1} < \ell_t \text{ and } \theta_{t+1} > \theta_t \text{ if } \sigma < 1 \\ \ell_{t+1} > \ell_t \text{ and } \theta_{t+1} < \theta_t \text{ if } \sigma > 1 \end{array} \right\}. \tag{39}$$

Proof. Expression (38) follows from setting $\bar{h} = 0$ in (35) and substituting (22) and (26). Result (39) follows from Proposition 1 combined with (31) that shows that, with $\bar{h} = 0$, θ_t is decreasing in ℓ_t. For κ_{ss} to be stable and unique, the elasticity (37) evaluated in κ_{ss} must be less than unity. Inserting $\kappa_t = \kappa_{t+1} = \kappa_{ss}$ and $\Gamma = 1$ and $\Gamma' = 0$ in (37), the elasticity reduces to

$$\frac{\mathrm{d}\kappa_{t+1}}{\mathrm{d}\kappa_t} = \alpha - \alpha \frac{\ell'_\kappa(\kappa_{ss})\kappa_{ss}}{\ell(\kappa_{ss})},$$

where the right hand side is less than unity if and only if

$$m_1(\kappa_{ss}) \equiv -\frac{\ell'_\kappa(\kappa_{ss})\kappa_{ss}}{\ell(\kappa_{ss})} \frac{\alpha}{1-\alpha} < 1. \tag{40}$$

In the Appendix we show that Assumption 1 is a sufficient condition for (40) to be satisfied. In the Appendix we also prove existence. ∎

Proposition 3 suggests three remarks. First, the dynamic law for capital (38) shows that, when there is no minimum requirement, investment per-young is proportional to the price of care. This is because savings only depend on current wages (w_t is proportional to p_t in each period). Second, given that capital per worker grows monotonically, both the wage and the price of care increase over time. Employment shares, however, move in opposite directions depending on the value of σ, which determines whether old agents increase or decrease their expenditure share on old-age care in response to increasing prices. The third remark is that, under complementarity, the savings rate θ_t increases during the transition because rising care prices attract labor in the care sector and the income share of young agents then grows – i.e., the intergenerational distribution effect.

The steady-state implications of the intergenerational distribution effect is immediate by comparing the steady-state level of the capital stock, κ_{ss}, with that in the canonical version of the model, which we term $\kappa_{ss}^{\text{canonical}}$. Starting from (34), and imposing $\bar{h} = 0$ and $\kappa_{t+1} = \kappa_t = \kappa_{ss}$, we obtain

$$\kappa_{ss} = \frac{1}{\ell(\kappa_{ss})^{\frac{\alpha}{1-\alpha}}} \left[\frac{B\beta(1-\alpha)}{(1+\beta)(1+n)} \right]^{\frac{1}{1-\alpha}} = \frac{1}{\ell(\kappa_{ss})^{\frac{\alpha}{1-\alpha}}} \kappa_{ss}^{\text{canonical}}, \tag{41}$$

where the steady-state level of capital per worker in the canonical model (which is obtained by setting $\ell_t = 1$ in each period) is

$$\kappa_{ss}^{\text{canonical}} = \left[\frac{B\beta(1-\alpha)}{(1+\beta)(1+n)} \right]^{\frac{1}{1-\alpha}}. \tag{42}$$

It immediately follows that $\kappa_{ss} > \kappa_{ss}^{\text{canonical}}$ always holds (as $\ell(\kappa_{ss}) < 1$), i.e., capital per worker in our model is higher than in the canonical model independently of whether generic goods and

care are complements or substitutes. The need for care increases the demand for labor, pushing income distribution in favor of the young, and therefore increases savings. (The size of the gap between κ_{ss} and $\kappa_{ss}^{\text{canonical}}$ depends, obviously, on the elasticity of substitution as well as the other parameters of the model through the term $\ell(\kappa_{ss})$, which we return to below).

4.3 Dynamics with Minimum Care Requirement

When the minimum old-age care requirement is strictly positive, $\bar{h} > 0$, the accumulation law (34) includes the dependency of current savings on future employment shares, i.e. the old-age requirement effect. This dynamic law determines the steady state(s) of the system and the associated stability properties. Under substitutability there is always a unique steady state. Under complementarity, i.e. $\sigma < 1$, we again in the main text assume that the elasticity of capital in generic production is not too high, now that is:[27]

Assumption 2: $\alpha < \frac{1-\alpha}{1-\sigma}$.

This assumption is sufficient (but not necessary) for the steady state to be unique. We then have:

Proposition 4 *Under Assumption 2 equation (35) exhibits a unique steady state $\bar{\kappa}_{ss}$ that is globally stable. The transitional dynamics of $p(\kappa_t)$ and $\ell(\kappa_t)$ comply with Proposition 1.*

Proof. For $\bar{\kappa}_{ss}$ to be stable and unique, the elasticity (37) evaluated in $\bar{\kappa}_{ss}$ must be less than unity. Inserting $\kappa_t = \kappa_{t+1} = \bar{\kappa}_{ss}$ in (37), the elasticity reduces to

$$\frac{\mathrm{d}\kappa_{t+1}}{\mathrm{d}\kappa_t} = \frac{\alpha - \alpha \frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}}{1 + \frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}},$$

where the right hand side is less than unity if and only if

$$m_1(\bar{\kappa}_{ss}) + m_2(\bar{\kappa}_{ss}) < 1, \tag{43}$$

with

$$m_2(\bar{\kappa}_{ss}) \equiv -\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{1}{1-\alpha}\begin{cases} < 1 \text{ if } \sigma < 1 \\ < 0 \text{ if } \sigma > 1 \end{cases}. \tag{44}$$

In the the Appendix we show that Assumption 2 is a sufficient condition for (43) to be satisfied. ∎

Proposition 4 establishes that, also with a minimum old-age care requirement, under complementarity the savings rate increases during a transition path where capital grows. Now, this is the combined result of the future-requirement and intergenerational distribution effects.

Imposing $\kappa_{t+1} = \kappa_t = \bar{\kappa}_{ss}$ in (34), the steady-state level of capital per worker must satisfy

$$\bar{\kappa}_{ss} = \Gamma\left(\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\right)^{\frac{1}{1-\alpha}}\frac{1}{\ell(\bar{\kappa}_{ss})^{\frac{\alpha}{1-\alpha}}} \cdot \kappa_{ss}^{\text{canonical}}. \tag{45}$$

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[27]Note that even in the limiting case where $\sigma \to 0$ this assumption is satisfied with the empirically plausible restriction $\alpha < \frac{1}{2}$. In the Appendix part B, we solve the model for the case in which Assumption 2 is not satisfied.

Comparing (45) to (41), since Γ strictly exceeds one when $\bar{h} > 0$, we thus conclude that $\bar{\kappa}_{ss} > \kappa_{ss} > \kappa_{ss}^{\text{canonical}}$; the long-run level of capital per worker is higher when there is a positive minimum requirement of old-age care, which drives capital further above the level attained in the canonical model. The reason is the minimum-requirement effect, which prompts households to save more during the transition in response to the continuous increase of the price of old-age care.

5 The Savings Multiplier

This section clarifies further how the intergenerational distribution effect and the old-age requirement effect give rise to a *savings multiplier*, where savings and capital accumulation stimulates further savings and capital accumulation.[28] Subsection 5.2, by investigating the effects of increased productivity, introduces and discusses the savings multiplier. Subsection 5.2 performs comparative-statics exercises suggesting that reduced population growth, e.g. because of the Chinese one-child policy, may boost capital accumulation via the savings multiplier. Subsection 5.3 points out that related effects may result from agents needing to purchase more health care through the market, e.g. because of modernization of society where the possibilities for family based care decreases.

5.1 Increased productivity

In this subsection we develop the savings multiplier by considering variations in the productivity level B, and we focus on the case of complementarity, $\sigma < 1$, which seems the most interesting, and many would claim, the most empirically relevant scenario.[29] The effects of exogenous shocks on income per capita may, as we will see in this subsection and the two next, differ substantially from those predicted by the canonical model. For expositional clarity, we start out without the minimum requirement effect, before we extend the analysis to include this.

Zero Requirement. In the canonical model, an exogenous increase in productivity increases the long-run level of (log) capital per worker in (42) by

$$\frac{\mathrm{d}\log \kappa_{ss}^{\text{canonical}}}{\mathrm{d}B} = \frac{1}{B(1-\alpha)}. \tag{46}$$

Now consider our model without minimum requirement. With $\bar{h} = 0$, the steady-state capital per worker is κ_{ss} defined in (41), and the impact of the productivity shock is determined by

$$\frac{\mathrm{d}\log \kappa_{ss}}{\mathrm{d}B} = \underbrace{\frac{1}{1-m_1(\kappa_{ss})}}_{\text{Savings Multiplier}} \left(\frac{\mathrm{d}\log \kappa_{ss}^{\text{canonical}}}{\mathrm{d}B} + m_1(\kappa_{ss}) \frac{\ell_B'(\kappa_{ss})}{\ell_\kappa'(\kappa_{ss}) \cdot \kappa_{ss}} \right), \tag{47}$$

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[28] Naturally, the convergence of this multiplier process is guaranteed exactly when the steady state is unique and stable.

[29] However, all of the equations to follow are identical also in the case of $\sigma > 1$, the only difference being in the qualitative strength of the effects. As will be easily understood below, all savings multipliers which exceed one when $\sigma < 1$, falls short of one when $\sigma > 1$. Thus shocks that are magnified with complementarity, are instead dampened with substitutability.

The crucial element in (47) is *the savings multiplier*, where m_1 is already defined in (40). Focusing on the case of complementarity, m_1 is strictly positive, and is less than unity in view of the stability of the steady state.[30] Since $0 < m_1 < 1$, the savings multiplier in (47) is strictly higher than unity. Combining this result with $\ell'_\kappa < 0$ and $\ell'_B < 0$,[31] we conclude that the impact of a productivity shock on steady-state capital per worker is stronger than that predicted by the canonical model. There are two reasons for this, both related to the intergenerational distribution effect. The first reason, which is the result of the intergenerational distribution effect in the static part of the model, is represented by the term $m_1 \frac{\ell'_B}{\ell'_\kappa \kappa} > 0$. The productivity increase pushes labor into care and out of generic production, increasing the wage further as compared to the canonical model, shifting income distribution in favor of the young. This means that the initial increase in the savings rate as a result of better productivity is higher than in the canonical model. The second reason, which is the result of the intergenerational distribution effect in the dynamic part of the model, is represented by the savings multiplier; the term $\frac{1}{1-m_1} > 1$. In our model, as the capital stock starts to grow, this further pushes labor out of generic production and into care, increasing the wage even further, thus magnifying the initial increase in savings. The implication is that a higher productivity increases the capital stock and wages by more than in the canonical model. As we will see below, the savings multiplier is also part of the explanation why low population growth and one-child policies may have such a massive impact on savings and capital accumulation.

Positive Requirement. To see how the old-age requirement $\bar{h} > 0$ modifies the savings multiplier, we again investigate the response of the steady-state capital stock to increased productivity, which from (45) is now given by

$$\frac{\mathrm{d} \log \bar{\kappa}_{ss}}{\mathrm{d}B} = \underbrace{\frac{1}{1 - [m_1(\bar{\kappa}_{ss}) + m_2(\bar{\kappa}_{ss})]}}_{\text{Savings multiplier}} \left[\frac{\mathrm{d} \log \kappa_{ss}^{\text{canonical}}}{\mathrm{d}B} + \frac{(m_1(\bar{\kappa}_{ss}) + m_2(\bar{\kappa}_{ss})) \ell'_B(\bar{\kappa}_{ss})}{\ell'_\kappa(\bar{\kappa}_{ss}) \bar{\kappa}_{ss}} \right], \quad (48)$$

where m_2 is defined in (44). Focusing again on the case of complementarity, the term $m_1 + m_2$ is strictly positive, and is less than unity in view of the stability of the steady state.[32] Since $0 < m_1 + m_2 < 1$, the savings multiplier in (48) is strictly higher than unity.

Compared to the case with $\bar{h} = 0$ in (47), the effect of increased productivity on steady-state capital now involves two additional effects strengthening the impact of productivity on steady state capital. These are identified by the two appearances of the term m_2 in (48). First, in the static equilibrium of the model, the higher wage now also means higher cost of old-age minimum requirement of care, implying an additional increase in savings compared to in the case above.

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[30] Under complementarity, m_1 is positive because $\ell'_\kappa < 0$ – see expression (36) – and is strictly less than unity in view of the stability condition proven in Proposition 3. Under substitutability, instead, expression (36) implies $\ell'_\kappa > 0$ and therefore $m_1 < 0$.

[31] Under complementarity, $\ell'_\kappa < 0$ follows from (36) whereas $\ell'_B < 0$ is established in Proposition 2.

[32] Under complementarity, both m_1 and m_2 are positive because $\ell'_\kappa < 0$ – see expression (36) – and $m_1 + m_2$ is strictly less than unity in view of the stability condition (43) proven in Proposition 4. Under substitutability, instead, expression (36) implies $\ell'_\kappa > 0$ and therefore $m_1 + m_2 < 0$, which yields a savings multiplier below unity.

Second, the savings multiplier increases, strengthening the feedback of capital on capital growth: the increase in the capital stock makes the wage rise over time, increasing the cost of the future minimum requirement, in turn increasing savings and the capital stock even more as compared to the case with $\bar{h} = 0$. Thus, both the static and dynamic effects generated by the old-age requirement effect reinforce the steady-state response of capital to increased productivity.

5.2 Population Growth

As is well known from the canonical model, a lower growth rate of population increases the steady-state level of capital per worker: from (42), we find

$$\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{-\mathrm{d}n} = \frac{1}{(1+n)(1-\alpha)} > 0. \tag{49}$$

In contrast, from (45), and taking into account (29), we find that the effect in our extended model with minimum requirement, $\bar{h} > 0$, is given by

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{-\mathrm{d}n} = \frac{1}{1-[m_1+m_2]} \left[\frac{\mathrm{d}\log\kappa_{ss}^{\text{canonical}}}{-\mathrm{d}n} + \frac{\ell_n'}{(-\ell_\kappa')\bar{\kappa}_{ss}}(m_1+m_2) + \frac{\ell}{(1+n)(-\ell_\kappa')\bar{\kappa}_{ss}}m_2 \right], \tag{50}$$

where we suppress the argument $\bar{\kappa}_{ss}$ to simplify the notation.[33] Assuming again complementarity, $\sigma < 1$, the multiplier $\frac{1}{1-m_1-m_2}$ is positive and higher than unity. There are, thus, five reasons the relative increase in capital with a lower growth of population is higher as compared to in the canonical model. The first two effects arise because of the intergenerational distribution effect and the old-age requirement effect, which through the savings multiplier ensures that capital growth has a stronger positive feedback on itself. These two effects are represented by m_1 and m_2 in the savings multiplier. The third and fourth effects are represented by the presence of m_1 and m_2 in the term $\frac{\ell_n'}{(-\ell_\kappa')\bar{\kappa}_{ss}}(m_1+m_2)$ in (50). They represent the effects on the labor share in the static model. With lower population growth there are fewer young relative to old agents at each point in time. This pull workers out of generic production and into care, increasing the wage. The increased wage increases the aggregate savings rate through both the intergenerational distribution effect and the old-age requirement effect. The fifth effect is represented by the term $\frac{\ell}{(1+n)(-\ell_\kappa')\bar{\kappa}_{ss}}m_2$ in (50). At each point in time, there is a higher fraction of old-age to young-age agents. Even for a fixed labor allocation this increases the wage. Through the old-age requirement effect, this increases the savings rate of the young in the static model even more, stimulating capital accumulation further. In total, this means that the effect of population growth in the present model may be substantially magnified compared to standard OLG models.

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[33]In (50), the terms $m_1, m_2, \ell, \ell_n', \ell_\kappa'$ are all evaluated in the steady state $\bar{\kappa}_{ss}$. Also, in deriving (50), we exploit the fact that $\frac{d\Gamma}{dn} = -\Gamma'\frac{\bar{h}}{(1+n)\ell}$ from expression (32). See the Appendix for a full derivation.

5.3 Increased need for health care

In the model, increased need to buy health care through the market may can be represented by a higher $\bar{h}$. Obviously, this draws resources out of generic sector production and into the production of care. The effect on steady-state capital is found by (45) to be

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}} = \frac{\frac{\ell'_{\bar{h}}}{\ell'_{\kappa}\cdot\bar{\kappa}_{ss}}(m_1+m_2) - \frac{\ell}{\bar{h}\cdot\ell'_{\kappa}\bar{\kappa}_{ss}}m_2}{1-(m_1+m_2)}. \tag{51}$$

A higher minimum requirement of care has a direct positive effect on savings, represented by the term $-\frac{\ell}{\bar{h}\ell'_{\kappa}\bar{\kappa}_{ss}}m_2$. In addition, the demand for labor increases, pushing the wage up. This shifts income distribution in favor of the young, and also makes care more expensive. For both reasons, savings increase, represented by the term $\frac{\ell'_{\bar{h}}}{\ell'_{\kappa}\bar{\kappa}_{ss}}(m_1+m_2)$. Thus, through three channels, a higher minimum requirement increases savings in the static model. Stimulated by the savings multiplier, steady-state capital increases by more than the immediate effect on savings and capital accumulation. As a consequence, the increased need for market based care may have a strong positive impact on capital accumulation.

6 Extensions

In this section we first extend the model to study social security in the form of a pay as you go system where the government provides old-age care. We then consider the possibility of dynamic inefficiency. We lastly extend the model to study endogenous growth dynamics. To put the main focus on the new effects our extensions introduce, throughout this section we for simplicity focus on the case where $\bar{h} = 0$.

6.1 Introducing the Welfare State

The savings motive in our model comes partly from the need to pay for future old-age care. It may be argued that this motive is particularly relevant for China, since the one-child policy and the missing welfare system implies that there are few other options. As argued by e.g. Oksanen (2010, p. 14), even in areas where the one-child policy has been less strictly enforced it has main implications: "Although this policy has been more relaxed in rural areas, the migration of descendants to cities has meant that many elderly people in rural areas are left without sufficient family support. ... .as the state induced the decline in fertility by regulation, the state must take responsibility, both financially and otherwise, for support for the elderly who are left with only narrow or no family based security". By extending our model, we can study some potential effects of the introduction of such a welfare state. In this subsection we consider the consequences of adopting a pay as you go scheme where the young pay a proportional tax τ_t on their income so as to finance free care g_t to the old living in the same period.[34] A balanced

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[34]Whether the tax income is used to provide care, or is transferred lump-sum from the young to the old, has no bearing our results (as long as the publicly provided care falls short of what old agents would purchase by themselves in absence of publicly provided care).

budget then requires that $\tau_t w_t (1+n) = p_t g_t$, which from (12) is equivalent to

$$\tau_t \eta (1+n) = g_t \tag{52}$$

While (22) is unaffected by the tax, we show in the Appendix that (24) is now modified to

$$p_t = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\sigma}{\sigma-1}} \left[\frac{(1-\ell_t)(1-\alpha)}{\ell_t - (1-\alpha)(1-\tau_t)}\right]^{\frac{1}{1-\sigma}} \tag{53}$$

In the static part of the model, it can thus easily be verified that a higher tax rate reduces employment in the generic sector (and increases employment in the production of care services). A further implication of this, in the static model, is that the decreased generic sector employment increases the wage.

In the Appendix we show that capital accumulation is now given by

$$\kappa_{t+1} = (1-\tau_t)\frac{B\beta(1-\alpha)}{(1+\beta)(1+n)}\kappa_t^\alpha \ell_t^{-\alpha}\left[1 + \frac{(1-\alpha)\tau_{t+1}}{\alpha(1+\beta)\ell_{t+1}}\right]^{-1} \tag{54}$$

The accumulation is affected through decreased savings as current wage income is now taxed. This is captured by the term $1 - \tau_t$, and is the same as in the canonical model. In addition, in the present model the tax also affects capital accumulation through the terms $\ell_t = \ell(\kappa_t, \tau_t)$ and $\ell_{t+1} = \ell(\kappa_{t+1}, \tau_{t+1})$, that now depend on the tax rates.

The total effect of taxes on the steady-state level of capital is found by imposing $\kappa_{t+1} = \kappa_t = \kappa_{ss}$, $\tau_{t+1} = \tau_t = \tau$, and taking into account that generic sector employment is decreasing in the tax rate through the term $\ell'_\tau \equiv d\ell(\kappa,\tau)/d\tau$ (for a given capital stock). We then find from (54) and (53) that κ_{ss} decreases when a tax financed transfer to the old is introduced.

$$\frac{d\kappa_{ss}}{d\tau} < 0$$

The proof of the sign is given in the Appendix. A higher tax rate and publicly provided old-age care redistributes income from the young to the old, and decreases the incentives to save. The higher labor demand in the care sector, which increases the wage for the young, cannot be sufficiently strong to turn this around. Thus, introducing the welfare state in e.g. China may remove parts in the engine that has fueled massive capital accumulation, and may have a strong negative impact on growth.

As is well known, however, a lower capital stock in an OLG framework need not reduce welfare. Thus, we next discuss dynamic inefficiency.

6.2 Dynamic Inefficiency and Golden Rule

Dynamic inefficiency is characterized by a situation where the steady state capital stock is so high that it is possible to consume part of it with no generation becoming worse off. As is well known, this is the case if the capital stock is above the what is often termed the "golden rule" capital stock. The golden rule capital stock, κ^*, is found by letting a social planner optimize with

respect to κ, ℓ, d, h, and c, finding $(\kappa^*, \ell^*, d^*, h^*, c^*)$ such that the consumer in each generation gets equal and maximum utility. We can, as all generations are considered equal from the social planer's point of view, suppress time subscripts. This implies maximizing

$$U \equiv u(c) + \beta v(d, h),$$

under the constraints

$$
\begin{aligned}
x &= B\kappa^\alpha \ell^{1-\alpha} = c + \frac{d}{1+n} + \kappa(1+n), \\
1 &= \ell + \frac{h}{\eta(1+n)}.
\end{aligned}
$$

The first constraint states that production in one period should pay for consumption of the young, c, and of the old, d. In addition, the capital labor ratio in each period should be preserved for the next period. The second constraint limits the labor use to the available labor force.

The solution for κ can be found immediately. Optimality requires that, given the optimal choice of all other variables, there should be no gain from changing the value of a subset of variables. In particular, given (ℓ^*, d^*, h^*), there should be no scope for increasing c by altering κ. Hence in optimum

$$\frac{\partial x}{\partial \kappa} = (1+n) \Rightarrow \kappa^* = \ell^* \left(\frac{B\alpha}{1+n}\right)^{\frac{1}{1-\alpha}},$$

where $\ell^* < 1$ is the optimal generic sector employment fraction. The implications for the golden rule capital ratio of introducing the old-age care sector is then immediate by comparing κ^* with that in the canonical version of the model, which we term $\kappa_{ss}^{*\text{canonical}}$. In the canonical model $\ell^* = 1$, hence

$$\kappa^* = \ell^* \left(\frac{B\alpha}{1+n}\right)^{\frac{1}{1-\alpha}} = \ell^* \kappa^{*\text{canonical}} < \kappa^{*\text{canonical}}.$$

Therefore, as compared to the canonical model, our model with the service sector for old-age care increases the potential relevance of dynamic inefficiency for two reasons. First, from (41) we know that the care sector generates an income distribution effect that leads to a steady state capital stock higher in our model than in the canonical model. Second, as the service sector reduces the labor available for the generic sector, this lowers the golden rule capital stock. Thus the actual capital stock in our model is higher than in the canonical model, while the golden rule capital stock is lower. For parameter spaces where the canonical model is efficient, our model could very well be inefficient, and moreover for parameter spaces where the canonical model is inefficient, our model is even further away from efficiency.

6.3 Endogenous Growth

In this subsection, we study the model with learning by doing. We show that, under complementarity, accumulation is self-reinforcing because capital growth induces positive feedback

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effects on saving rates, and thereby subsequent accumulation. In contrast, under substitutability accumulation is self-balancing, as capital growth induces negative feedback effects on saving rates.

Substituting the learning-by-doing specification (11) in (17) and (33), the linear AK model yields the accumulation law

$$\frac{\kappa_{t+1}}{\kappa_t} = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}\frac{1}{\ell(\kappa_t)}.$$

(55)

Whether capital accumulation accelerates or converges to a stable pace depends on the co-movements of capital and employment shares. Recalling Proposition 1, we have $\ell'_\kappa < 0$ under complementarity, and $\ell'_\kappa > 0$ under substitutability. We now discuss the consequences of these processes for endogenous long-run growth.

Complementarity: Self-Reinforcing Accumulation and Traps

Considering first the case of complementarity, we may observe (aside from the very special case of permanent steady state discussed below) two types of growth paths:

(i) *Self-Reinforcing Accumulation.* Capital per worker and the price of health care grow forever. During the transition, the employment share of the generic-good sector declines and the saving rate grows. In the long run, the economy converges asymptotically to the equilibrium featuring

$$\lim_{t\to\infty}\frac{\kappa_{t+1}}{\kappa_t} = \frac{A\beta}{(1+\beta)(1+n)} > 1 \text{ and } \lim_{t\to\infty}\ell(\kappa_t) = 1-\alpha.$$

(56)

(ii) *Self-Reinforcing Decumulation.* Capital per worker, the price of health care and the saving rate decline over time while the employment share of the generic-good sector grows. In the long run, the economy converges asymptotically to the equilibrium featuring

$$\lim_{t\to\infty}\kappa_t = 0 \text{ and } \lim_{t\to\infty}\ell(\kappa_t) = 1.$$

(57)

Self-reinforcing accumulation results from the fact that, under complementarity, capital accumulation induces positive feedback effects on the savings rate. An initial increase in capital per worker drives up the health-care price and reduces the employment share of the generic sector: the intergenerational distribution effect then implies a higher saving rate, and thereby further capital accumulation. Symmetrically, an initial decline in capital per worker results in self-reinforcing decumulation via lower saving rates. Depending on initial endowments, the economy may undertake a permanent accumulation path, or remain trapped in a permanent decumulation path. The next Proposition defines the critical level of capital per worker at time zero, which acts as a threshold between the accumulation and decumulation outcomes.

Proposition 5 *(AK model under complementarity) If* $1-\alpha < \frac{(1+\beta)(1+n)}{A\beta} < 1$, *there exists a finite critical level* $\tilde{\kappa} > 0$ *satisfying*

$$\ell(\tilde{\kappa}) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)},$$

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and acting as a separating threshold: if $\kappa_0 > \tilde{\kappa}$ ($\kappa_0 < \tilde{\kappa}$), the economy follows the self-reinforcing accumulation (decumulation) path forever. If $\frac{(1+\beta)(1+n)}{A\beta} < 1 - \alpha < 1$, the economy follows the self-reinforcing accumulation path for any $\kappa_0 > 0$. If $1 - \alpha < 1 < \frac{(1+\beta)(1+n)}{A\beta}$, the economy follows the self-reinforcing decumulation path for any $\kappa_0 > 0$.

Proof. See Appendix. ∎

The intuition behind Proposition 5 can be seen as follows. When $\sigma < 1$, the equilibrium employment share $\ell(\kappa_t)$ is negatively related to κ_t. Therefore, if the initial stock is sufficiently high to satisfy $\kappa_0 > \tilde{\kappa}$, we have

$$\ell(\kappa_0) < \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)},$$

the economy exhibits positive capital growth in the first period, $\kappa_1 > \kappa_0$, the generic-sector employment share declines, $\ell(\kappa_1) < \ell(\kappa_0)$, and the growth rate of capital in the subsequent period is even higher, $\kappa_2/\kappa_1 > \kappa_1/\kappa_0$. This mechanism arises in all subsequent periods and drives the economy towards the asymptotic equilibrium described in expression (56) above.[35] Symmetrically, if the initial stock is relatively low, $\kappa_0 < \tilde{\kappa}$, the same self-reinforcing mechanism works in the opposite direction: generic-sector employment is initially so high that savings are discouraged and capital per worker declines, implying further increase in $\ell(\kappa)$ and therefore permanent decumulation. In the very special case $\kappa_0 = \tilde{\kappa}$, there is a permanent steady-state equilibrium: capital per worker and employment shares are constant forever. In this situation, however, any small perturbation affecting capital per-worker would drive the economy towards self-reinforcing accumulation or decumulation: see Figure 3, diagram (a), for a graphical description of this result.

Proposition 5 also shows that, if preference and technology parameters do not satisfy $1 - \alpha < \frac{(1+\beta)(1+n)}{A\beta} < 1$, only one of the two paths survives: there is either self-reinforcing accumulation or self-reinforcing decumulation because the critical threshold on capital per worker $\tilde{\kappa}$ cannot be positive and finite.[36]

Substitutability: Self-Balancing Accumulation and Stagnation

Considering next the case of substitutability, the analysis of equation (55) is modified by the fact that $\ell'_\kappa > 0$. When $\sigma > 1$, the generic-sector employment share increases with capital because old agents respond to higher health-care prices by spending a higher fraction of income on generic goods. This implies that, contrary to the case of complementarity, an initial increase in capital generates *negative feedback effects* on savings through the intergenerational distribution effect: higher generic-sector employment reduces the total income share of young agents and, hence, the economy's saving rate. The consequences for economic growth are summarized in the following proposition:

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[35]Expressions (56) represent an asymptotic equilibrium that is never reached in finite time. The proof follows from our previous analysis in Figure 2: as κ grows forever, the curve $\Phi(\ell, \kappa)$ permanently shifts upward and the resulting equilibrium share $\ell(\kappa)$ approaches $1 - \alpha$ only asymptotically because $\lim_{\ell \to 1-\alpha^+} \Psi(\ell) = +\infty$.

[36]See the proof of Proposition 5 for details.

Proposition 6 *(AK model under substitutability) If $1 - \alpha < \frac{(1+\beta)(1+n)}{A\beta} < 1$, there exists a finite critical level $\hat{\kappa} > 0$ satisfying*

$$\ell(\hat{\kappa}) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)},$$

and representing a global attractor: if $\kappa_0 < \hat{\kappa}$ ($\kappa_0 > \hat{\kappa}$), the economy follows a self-balancing accumulation (decumulation) path during the transition, and converges from below (above) to the stationary long-run equilibrium featuring $\lim_{t\to\infty} \kappa_t = \hat{\kappa}$ and $\lim_{t\to\infty} \ell(\kappa_t) = \ell(\hat{\kappa})$. If $\frac{(1+\beta)(1+n)}{A\beta} < 1 - \alpha < 1$, the economy exhibits positive growth of κ_t forever and $\lim_{t\to\infty} \ell(\kappa_t) = 1$. If $1 - \alpha < 1 < \frac{(1+\beta)(1+n)}{A\beta}$, the economy exhibits negative growth of κ_t forever and $\lim_{t\to\infty} \ell(\kappa_t) = 1 - \alpha$.

Proof. See Appendix. ∎

The first result established in Proposition 6 may be restated as follows: when generic consumption and old-specific goods are strict substitutes, then if there exists a steady-state level of capital per worker that is compatible with positive production in both sectors, *the linear AK model behaves similarly to a neoclassical model.* Starting from relatively low capital, capital per worker grows over time but at decreasing rates, until the economy reaches a stable steady state representing the long-run equilibrium. However, this result is not due to decreasing returns to capital in production: differently from the neoclassical model, the convergence towards $\hat{\kappa}$ is determined by the reaction of sectoral employment shares to capital accumulation. Since capital growth increases employment in the generic sector, accumulation under substitutability is self-balancing. This conclusion is opposite to that obtained under complementarity, where accumulation is self-reinforcing. Figure 3, diagram (b), describes this result in graphical terms.

Proposition 6 also establishes the conditions under which substitutability admits permanent positive accumulation: when $\frac{(1+\beta)(1+n)}{A\beta} < 1 - \alpha < 1$, there is no finite steady state $\hat{\kappa}$ and the economy grows forever. However, the transitional dynamics of employment shares, saving rates and growth rates are qualitatively opposite to the case of complementarity: workers flow to the generic sector, the saving rate declines and growth decelerates.

Propositions 5 and 6 show that the elasticity of substitution between generic goods and health care bears fundamental implications for economic growth. On the one hand, the separating threshold level $\tilde{\kappa}$ that arises under complementarity recalls several conclusions of the literature on poverty traps in endogenous growth models – such as those going back to Azariadis and Drazen (1990) – but in our model this hinges on the degree of substitutability between generic consumption goods and old-specific consumption goods. Under substitutability, on the other hand, the linear AK model of endogenous growth, by generating a stable long-run equilibrium with stationary capital per worker $\hat{\kappa}$, shares fundamental properties with the neoclassical model. This is, to the best of our knowledge, a novel result. The fact that the steady-state levels of capital per worker, $\tilde{\kappa}$ and $\hat{\kappa}$, have opposite characteristics is conceptually linked to the results of Peretto and Valente (2011), who study the existence of pseudo-Malthusian equilibria in a
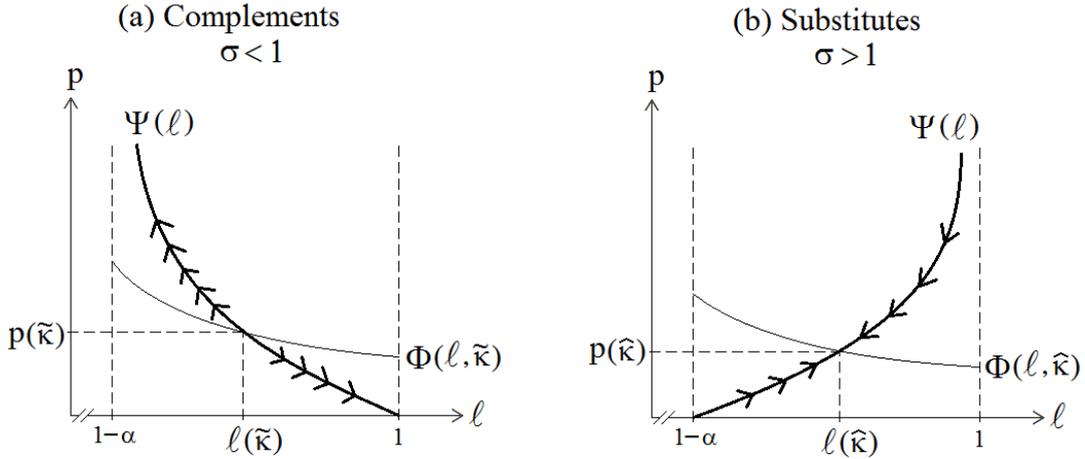
Figure 3: Dynamics of the AK model ($\bar{h} = 0$). The steady state of capital per worker is a separating threshold under complementarity, a global attractor under substitutability.

growth model with endogenous technological progress.[37]

7   Concluding Remarks

In this paper, we introduced the concept of savings multiplier, a general equilibrium mechanism that induces rising saving rates over time and that magnifies the impact of exogenous shocks on capital per capita in the long run. In our theory, capital accumulation affects savings via two channels. Real wages increase as the capital stock grows at the same time as workers move from the manufacturing sector to the labor-intensive service sector, implying a shift of the income distribution in favor of young savers relative to old agents (*intergenerational distribution effect*). Meanwhile, growth in real wages raises the anticipated cost of providing for the old age, prompting the currently young to save a higher fraction current income (*old-age requirement effect*). Both these mechanisms determine a positive feedback of capital accumulation on future accumulation, which implies circular causality between economic growth and saving rates over time.

Our theory provides a new explanation for rising saving rates in developing countries and, more specifically, is consistent with the stylized facts that characterize China's economic performance. Studying exogenous shocks that plausibly capture the effects of China's past reforms, we argue that the one-child policy and the dismantling of cradle-to-grave social benefits have fuelled China's saving rates in the past decades. The model also predicts that introducing a new

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[37] Peretto and Valente (2011) show that population size – somewhat similarly to the variable "capital per worker" in our model – exhibits an unstable steady state when labor and land are complements, and a stable steady state when inputs are substitutes. Besides the totally different aim of the analysis, the implications of these results for economic growth are opposite in the two models: in our framework, growing capital is necessary for output growth whereas stationary population in Peretto and Valente (2011) induces permanent economic growth in income per capita generated by innovations.

social security system, as currently discussed by the chinese authorities, would imply reduced saving rates and lower capital per capita in the long run, but not necessarily lower individual welfare. Infact, the mechanism of the savings multiplier implies that past reforms may have created overaccumulation (i.e., dynamic inefficiency), in which case implementing a social security system that reduces saving rates turns out to be welfare-improving.

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A   Appendix for Reviewers

Consumption levels: derivation of (18)-(19). The household maximizes (2) subject to (4)-(5). The Lagrangean at time t reads

$$\mathcal{L} \equiv u\left(c_t\right) + \beta v\left(d_{t+1}, h_{t+1} - \bar{h}\right) + \lambda_1 \left(w_t - s_t - c_t\right) + \lambda_2 \left(s_t R_{t+1} - d_{t+1} - p_{t+1}^h h_{t+1}\right)$$

where λ_1 and λ_2 are the multipliers. The first-order conditions with respect to $(c_t, d_{t+1}, h_{t+1}, s_t)$ are

$$u_{c_t}\left(c_t\right) = \lambda_1, \tag{A.1}$$

$$\beta v_{d_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right) = \lambda_2, \tag{A.2}$$

$$\beta v_{h_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right) = \lambda_2 p_{t+1}, \tag{A.3}$$

$$\lambda_1 = \lambda_2 R_{t+1}. \tag{A.4}$$

Combining (A.1) with (A.2) and (A.2) with (A.3), we respectively obtain

$$u_{c_t}\left(c_t\right) = \beta R_{t+1} v_{d_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right), \tag{A.5}$$

$$v_{h_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right) = p_{t+1} v_{d_{t+1}}\left(d_{t+1}, h_{t+1} - \bar{h}\right), \tag{A.6}$$

where (A.5) is the Keynes-Ramsey rule for the generic good, and (A.6) equates the relative price of health care services to the marginal rate of substitution with second-period consumption. Exploiting the assumed utility functions (6)-(7), conditions (A.5)-(A.6) respectively read

$$d_{t+1} = c_t \beta R_{t+1} - \left(h_{t+1} - \bar{h}\right) \frac{1-\gamma}{\gamma}\left(\frac{d_{t+1}}{h_{t+1} - \bar{h}}\right)^{\frac{1}{\sigma}}, \tag{A.7}$$

$$p_{t+1} = \frac{1-\gamma}{\gamma}\left(\frac{d_{t+1}}{h_{t+1} - \bar{h}}\right)^{\frac{1}{\sigma}}. \tag{A.8}$$

Substituting (A.8) in (A.7) gives

$$d_{t+1} + p_{t+1}\left(h_{t+1} - \bar{h}\right) = c_t \beta R_{t+1}. \tag{A.9}$$

Substituting (A.9) in the second-period budget constraint (5) and then using the first-period budget constraint (4) to eliminate savings, we obtain expression (18) in the text. Next, substitute the market clearing condition $K_{t+1} = N_t^y s_t$ into (5) to obtain

$$R_{t+1} K_{t+1} = N_t^y \left(d_{t+1} + p_{t+1} h_{t+1}\right). \tag{A.10}$$

Given the market clearing condition $N_t^o h_t = H_t$, the zero-profit condition for the health care sector reads

$$p_t N_t^o h_t = w_t \left(1 - \ell_t\right) N_t^y. \tag{A.11}$$

Substituting (A.11) for period $t + 1$ into (A.10), we obtain

$$N_{t+1}^o d_{t+1} = R_{t+1} K_{t+1} - w_{t+1} (1 - \ell_{t+1}) N_{t+1}^y. \tag{A.12}$$

From the profit maximizing conditions (16) and (15), respectively, we have

$$R_t K_t = B\alpha \left[N_t^y (\kappa_t)^\alpha (\ell_t)^{1-\alpha} \right], \tag{A.13}$$

$$w_t (1 - \ell_t) N_t^y = B (1 - \alpha) \frac{1 - \ell_t}{\ell_t} \left[N_t^y (\kappa_t)^\alpha (\ell_t)^{1-\alpha} \right]. \tag{A.14}$$

Setting (A.12) at period t and substituting (A.13) and (A.14), we obtain equation (19) in the text. Note that result (19) implies a restriction: second-period generic consumption is positive if and only if $\ell_t > 1 - \alpha$.

Consumer problem: derivation of (20). Setting expression (A.8) at time t, raising both sides to the power of σ, and dividing both sides by p_t, we obtain expression (20) in the text.

Goods Market: derivation of (24). Starting from (20), multiply both d_t and $\left(h_t - \bar{h} \right)$ by old population N_t^o, and substitute the old agents' constraint $N_t^o d_t = N_t^o s_{t-1} R_t - p_t H_t$, to obtain

$$p_t^{\sigma-1} = \left(\frac{1 - \gamma}{\gamma} \right)^\sigma \frac{N_t^o s_{t-1} R_t - p_t H_t}{p_t H_t - p_t N_t^o \bar{h}}. \tag{A.15}$$

Substituting capital income with the profit-maximizing condition $N_t^o s_{t-1} R_t = \alpha X_t$, we get

$$p_t^{\sigma-1} = \left(\frac{1 - \gamma}{\gamma} \right)^\sigma \frac{\alpha X_t - p_t H_t}{p_t H_t - p_t N_t^o \bar{h}}. \tag{A.16}$$

Recalling that constant returns to scale in both production sectors imply the zero profit conditions

$$X_t = N_t^y (w_t \ell_t + R_t \kappa_t),$$
$$p_t H_t = N_t^y w_t (1 - \ell_t),$$
$$\text{and } \frac{R_t \kappa_t}{w_t} = \frac{\alpha}{1-\alpha} \ell_t,$$

expression (A.16) reduces to

$$p_t^{\sigma-1} = \left(\frac{1 - \gamma}{\gamma} \right)^\sigma \frac{1}{1 - \alpha} \frac{N_t^y w_t}{p_t} \frac{\ell_t - (1 - \alpha)}{H_t - N_t^o \bar{h}}. \tag{A.17}$$

From (8) and the definition of $\ell^{\max}$, we have $H_t - N_t^o \bar{h} = \eta (\ell^{\max} - \ell_t) N_t^y$. Plugging this result into (A.17), and substituting $\frac{w_t}{\eta p_t} = 1$ from (12), we obtain expression (24) in the text.

Existence and uniqueness of the fixed point (25). The functions $\Phi (\ell_t, \kappa_t)$ defined in (21) exhibit the following properties:

$$\lim_{\ell_t \to 1-\alpha} \Phi (\ell_t, \kappa_t) = \left\{ \begin{array}{ll} (B/\eta) (1 - \alpha)^{1-\alpha} (\kappa_t)^\alpha & \text{(Neoclassical)} \\ (A/\eta) \kappa_t & \text{(Linear AK)} \end{array} \right\}$$

$$\lim_{\ell_t \to \ell^{\max}} \Phi (\ell_t, \kappa_t) = \left\{ \begin{array}{ll} (B/\eta) (1 - \alpha) (\kappa_t/\ell^{\max})^\alpha & \text{(Neoclassical)} \\ (A/\eta) (1 - \alpha) (\kappa_t/\ell^{\max}) & \text{(Linear AK)} \end{array} \right\}, \tag{A.18}$$

with derivatives

$$\Phi_{\ell_t} \equiv \frac{\partial \Phi(\ell_t, \kappa_t)}{\partial \ell_t} = \left\{ \begin{array}{ll} -\alpha \frac{\Phi(\ell_t, \kappa_t)}{\ell_t} < 0 & \text{(Neoclassical)} \\ -\frac{\Phi(\ell_t, \kappa_t)}{\ell_t} < 0 & \text{(Linear AK)} \end{array} \right\} \quad \text{and} \quad \Phi_{\ell_t \ell_t} \equiv \frac{\partial^2 \Phi(\ell_t, \kappa_t)}{\partial \ell_t^2} > 0. \quad \text{(A.19)}$$

The elasticity of $\Phi(\ell_t, \kappa_t)$ in the two cases is

$$\frac{\Phi_{\ell_t} \ell_t}{\Phi} = \left\{ \begin{array}{ll} -\alpha & \text{(Neoclassical)} \\ -1 & \text{(Linear AK)} \end{array} \right\} \quad \text{(A.20)}$$

The function defined in (24), instead, exhibits

$$\lim_{\ell_t \to 1-\alpha} \Psi(\ell_t) = \left\{ \begin{array}{llll} \infty & \text{if } \sigma < 1; & 0 & \text{if } \sigma > 1 \end{array} \right\},$$

$$\lim_{\ell_t \to \ell^{\max}} \Psi(\ell_t) = \left\{ \begin{array}{llll} 0 & \text{if } \sigma < 1; & \infty & \text{if } \sigma > 1 \end{array} \right\}, \quad \text{(A.21)}$$

with

$$\Psi'(\ell_t) \equiv \frac{\partial \Psi(\ell_t)}{\partial \ell_t} = \frac{\Psi(\ell_t)}{\sigma - 1} \frac{\ell^{\max} - (1-\alpha)}{(\ell^{\max} - \ell_t)[\ell_t - (1-\alpha)]} \left\{ \begin{array}{ll} < 0 & \text{if } \sigma < 1 \\ > 0 & \text{if } \sigma > 1 \end{array} \right\}. \quad \text{(A.22)}$$

The elasticity is therefore

$$\frac{\Psi'(\ell_t) \ell_t}{\Psi(\ell_t)} = -\frac{1}{1-\sigma} \underbrace{\frac{\ell^{\max} - (1-\alpha)}{\ell^{\max} - \ell_t}}_{>1} \underbrace{\frac{\ell_t}{\ell_t - (1-\alpha)}}_{>1}. \quad \text{(A.23)}$$

Under substitutability, existence and uniqueness of the fixed point (25) are guaranteed by the derivatives (A.19)-(A.22) along with the limits (A.18) and (A.21). Under complementarity, expression (A.20) implies $\Phi_{\ell_t} \ell_t / \Phi > -1$ whereas expression (A.23) implies $\Psi'(\ell_t) \ell_t / \Psi(\ell_t) < -1$. These values of elasticities imply existence and uniqueness of the fixed point (25) even with $\sigma < 1$ despite the fact that both $\Phi(\ell_t, \kappa_t)$ and $\Psi(\ell_t)$ are strictly decreasing. For future reference, note that the limiting properties of $\Phi(\ell_t, \kappa_t)$ and $\Psi(\ell_t)$ described in (A.18) and (A.21) imply

$$\lim_{\kappa \to 0} \ell(\kappa) = \left\{ \begin{array}{ll} \ell^{\max} & \text{if } \sigma < 1 \\ 1 - \alpha & \text{if } \sigma > 1 \end{array} \right\} \quad \text{and} \quad \lim_{\kappa \to \infty} \ell(\kappa) = \left\{ \begin{array}{ll} 1 - \alpha & \text{if } \sigma < 1 \\ \ell^{\max} & \text{if } \sigma > 1 \end{array} \right\}. \quad \text{(A.24)}$$

Neoclassical growth: elasticity of $\ell(\kappa)$, derivation of (36). Totally differentiating the fixed-point condition $\Psi(\ell(\kappa)) = \Phi(\ell(\kappa), \kappa)$, we obtain

$$\ell'_\kappa(\kappa) = \frac{\Phi_\kappa(\ell(\kappa), \kappa)}{\Psi'(\ell(\kappa)) - \Phi_\ell(\ell(\kappa), \kappa)}. \quad \text{(A.25)}$$

In the Neoclassical case, function $\Phi(\ell(\kappa), \kappa)$ exhibits the partial derivatives

$$\Phi_\kappa(\ell, \kappa) = \alpha \Phi(\ell, \kappa) / \kappa \quad \text{and} \quad \Phi_\ell(\ell, \kappa) = -\alpha \Phi(\ell, \kappa) / \ell. \quad \text{(A.26)}$$

Substituting (A.26) and the equilibrium condition $\Psi(\ell(\kappa)) = \Phi(\ell(\kappa), \kappa)$ in (A.25), we obtain

$$\frac{\ell'_\kappa(\kappa) \kappa}{\ell(\kappa)} = \frac{\alpha}{\alpha + \frac{\Psi'(\ell(\kappa))\ell(\kappa)}{\Psi(\ell(\kappa))}}. \quad \text{(A.27)}$$

3

Result (A.27) establishes a clear link between the elasticity of the generic-sector employment share $\ell(\kappa)$ to the capital stock κ and the elasticity of the price of health care $\Psi(\ell(\kappa))$ to the generic-sector employment share $\ell(\kappa)$. In particular, substituting (A.23) in (A.27), we have

$$\frac{\ell'_\kappa(\kappa)\,\kappa}{\ell(\kappa)} = \frac{1}{1 - \frac{1}{1-\sigma}\frac{1}{\alpha}\left\{\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell(\kappa)}\frac{\ell(\kappa)}{\ell(\kappa)-(1-\alpha)}\right\}}, \tag{A.28}$$

where the term in curly brackets equals $Q_1 \equiv \frac{\ell_t}{\ell_t-(1-\alpha)}\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell_t}$ in expression (36). The fact that $Q_1 > 1$ directly follows from the equilibrium requirement $1-\alpha < \ell_t < \ell^{\max}$, and it implies that

$$\begin{array}{ll} \frac{1}{1-\sigma}\frac{1}{\alpha}Q_1 > 1 & \text{if } 0 < \sigma < 1, \\ \frac{1}{1-\sigma}\frac{1}{\alpha}Q_1 < 0 & \text{if } \sigma > 1. \end{array} \tag{A.29}$$

Results (A.29) imply the signs reported in expression (36) in the text.

Proposition 3: Existence, Uniqueness and Stability. To prove existence, consider equation (38) and substitute $p(\kappa_t) = \Psi(\ell(\kappa_t))$ from (26), obtaining

$$\kappa_{t+1} = \frac{\eta\beta}{(1+n)(1+\beta)}\Psi(\ell(\kappa_t)). \tag{A.30}$$

The right-hand side of (A.30) is strictly increasing in κ_t: differentiation with respect to κ_t yields

$$\frac{\mathrm{d}\kappa_{t+1}}{\mathrm{d}\kappa_t} = \frac{\eta\beta}{(1+n)(1+\beta)}\Psi'(\ell(\kappa_t))\ell'_\kappa(\kappa_t) > 0, \tag{A.31}$$

where the positive sign derives from the fact that both $\Psi'(\ell_t)$ and $\ell'_\kappa(\kappa_t)$ are negative (positive) under complementarity (substitutability). From (A.22), (A.21) and (A.24), we have

$$\lim_{\kappa\to 0}\Psi'(\ell(\kappa)) = \infty \text{ and } \lim_{\kappa\to\infty}\Psi'(\ell(\kappa)) = 0 \tag{A.32}$$

under both complementarity and substitutability. Results (A.31) and (A.32) imply existence of at least one steady state $\kappa_{ss} = \frac{\eta\beta}{(1+n)(1+\beta)}\Psi(\ell(\kappa_{ss}))$. Moreover, if the elasticity condition (40) is valid for any κ, i.e.

$$-\frac{\ell'_\kappa(\kappa)\,\kappa}{\ell(\kappa)}\frac{\alpha}{1-\alpha} < 1, \tag{A.33}$$

then the steady state $\kappa = \kappa_{ss}$ is unique and globally stable. Under substitutability, inequality (A.33) is necessarily satisfied: when $\sigma > 1$, expression (36) implies $\ell'_\kappa(\kappa) > 0$ and therefore a strictly negative left hand side in (A.33). To study the case of complementarity, substitute the elasticity $\frac{\ell'_\kappa(\kappa)\kappa}{\ell(\kappa)}$ by means of expression (A.28), and rearrange terms, to rewrite condition (A.33) as

$$1 - \frac{(1-\sigma)(1-\ell(\kappa))(\ell(\kappa)-(1-\alpha))}{\ell(\kappa)} > \alpha. \tag{A.34}$$

Condition (A.34) implies a more restrictive requirement on parameters the lower is the left hand side. In this respect, the left hand side of (A.34) is strictly increasing in σ so that, all else equal, it reaches its smallest value when $\sigma = 0$. Letting $\sigma = 0$, the stability condition becomes

$$\left(\ell(\kappa) - \frac{1}{2}\right)^2 > \alpha - \frac{3}{4}, \tag{A.35}$$

4

which is surely satisfied when $\alpha < 3/4$. Therefore, a generously sufficient, not necessary condition for stability and uniqueness under complementarity is $\alpha < 3/4$. Given uniqueness and stability, as κ_t converges to κ_{ss} in the long run, both the price of health care $p_t = p(\kappa_t)$ and the employment share $\ell_t = \ell(\kappa_t)$ converge to constant levels. Results (A.31) and (A.32) guarantee that the transitional dynamics of κ_t are monotonic. The transitional dynamics of $p(\kappa_t)$ and $\ell_t = \ell(\kappa_t)$ then follow directly from Proposition 1. ∎

Proposition 4: Uniqueness and Stability. The existence of the steady state $\bar{\kappa}_{ss}$ is proved in Appendix B along with the discussion of possible multiple steady states. Given existence, the steady state $\kappa = \bar{\kappa}_{ss}$ is unique and globally stable if the elasticity condition (43) is valid for any κ, i.e.

$$m_1(\kappa) + m_2(\kappa) < 1, \tag{A.36}$$

where

$$m_1(\kappa) \equiv -\frac{\ell'_\kappa(\kappa)\,\kappa}{\ell(\kappa)}\frac{\alpha}{1-\alpha}, \tag{A.37}$$

$$m_2(\kappa) \equiv -\frac{\ell'_\kappa(\kappa)\,\kappa}{\ell(\kappa)}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\kappa)}\frac{1}{1-\alpha}. \tag{A.38}$$

Expression (A.37) follows from generalizing the definition of m_1 given in (40) whereas (A.38) follows from generalizing the definition of m_2 given in (44). The inequalities appearing in (44) are part of the following proof. First, consider substitutability. When $\sigma > 1$, both $m_1(\kappa)$ and $m_2(\kappa)$ are strictly negative because expression (36) implies $\ell'_\kappa(\kappa) > 0$ and expression (32) implies $\Gamma' > 0$. Therefore, condition (A.36) is necessarily satisfied when $\sigma > 1$. To study the case of complementarity, note that (32) implies

$$\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\kappa)} = \frac{\frac{(1-\alpha)}{\alpha(1+\beta)\eta(1+n)}\frac{\bar{h}}{\ell(\kappa)}}{1 - \frac{(1-\alpha)}{\alpha(1+\beta)\eta(1+n)}\frac{\bar{h}}{\ell(\kappa)}} = \frac{\frac{(1-\alpha)(1-\ell^{\max})}{\alpha(1+\beta)}}{\ell(\kappa) - \frac{(1-\alpha)(1-\ell^{\max})}{\alpha(1+\beta)}}, \tag{A.39}$$

where the last term follows from substituting $\bar{h} = \eta(1+n)(1-\ell^{\max})$ by definition (13). Defining the convenient parameter

$$q_1 \equiv \frac{(1-\alpha)(1-\ell^{\max})}{\alpha(1+\beta)} > 0, \tag{A.40}$$

we can substitute (A.39) in (A.38) and rewrite the stability condition (A.36) as

$$-\frac{\ell'(\kappa)\,\kappa}{\ell(\kappa)}\cdot\frac{\alpha}{1-\alpha} - \frac{\ell'(\kappa)\,\kappa}{\ell(\kappa)}\cdot\frac{q_1}{\ell(\kappa) - q_1}\cdot\frac{1}{1-\alpha} < 1. \tag{A.41}$$

Fro future reference, note that parameter q_1 is always strictly less than $1 - \alpha$.[38] This implies that $\ell(\kappa) > q_1$ holds in any interior equilibrium:

$$q_1 < 1 - \alpha < \ell(\kappa). \tag{A.42}$$

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[38] Because any interior equilibrium satisfies $(1-\alpha) < \ell(\kappa_t) < \ell^{\max}$, it is necessarily true that $\ell^{\max} > 1-\alpha-\alpha\beta$. Consequently, the factor $\frac{(1-\ell^{\max})}{\alpha(1+\beta)}$ is strictly less than unity and this, in turn, implies that $q_1 \equiv (1-\alpha)\frac{(1-\ell^{\max})}{\alpha(1+\beta)}$ is strictly less than $(1-\alpha)$.

Going back to (A.41), substituting $\frac{\ell'_\kappa(\kappa)\kappa}{\ell(\kappa)}$ by means of (A.28), the stability condition reduces to

$$\frac{1-\alpha}{\alpha(1-\sigma)}\left[\frac{\ell(\kappa)-q_1}{\ell(\kappa)-(1-\alpha)}\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell(\kappa)}\right] > 1. \tag{A.43}$$

From result (A.42), the term in square brackets in (A.43) is strictly greater than unity.[39] Therefore, a sufficient but not necessary condition for satisfying (A.43) is that $\frac{1-\alpha}{\alpha(1-\sigma)} > 1$, which is equivalent to Assumption 2 in the text. The conclusion is that, when $\sigma < 1$, satisfying Assumption 2 is sufficient to guarantee stability and uniqueness of the steady state. Also note that the stability condition under complementarity guarantees $m_2(\kappa) < 1$, as reported in expression (44).[40]

Derivation of (46)-(47) . Expression (46) directly follows from log-differentiating (42) with respect to B. In (41), instead, log-differentiation with respect to B yields

$$\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}B} = -\frac{\alpha}{1-\alpha}\left[\frac{\ell'_\kappa(\kappa_{ss})\kappa_{ss}}{\ell(\kappa_{ss})}\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}B} + \frac{\ell'_B(\kappa_{ss})}{\ell(\kappa_{ss})}\right] + \frac{\mathrm{d}\log\kappa_{ss}^{\mathrm{canonical}}}{\mathrm{d}B}, \tag{A.44}$$

where the term in square brackets is the chain derivative $\frac{\mathrm{d}\log\ell(\kappa_{ss})}{\mathrm{d}B}$, with $\ell'_B(\kappa_{ss})$ representing the static derivative $\frac{\mathrm{d}\ell(\kappa;B)}{\mathrm{d}B}$ defined in Proposition 2, evaluated in the steady state κ_{ss}. Substituting the definition of $m_1 \equiv -\frac{\alpha}{1-\alpha}\frac{\ell'_\kappa\cdot\kappa}{\ell}$ from (40) into (A.44), and rearranging terms, we obtain (47).

Derivation of (48). From definition (32), we have

$$\frac{\mathrm{d}}{\mathrm{d}B}\log\Gamma\left(\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\right) = -\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}B},$$

where both Γ and Γ' in the right hand side are evaluated in $(\bar{h}/\ell(\bar{\kappa}_{ss}))$. Therefore, log-differentiating (45) with respect to B yields

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}B} = -\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}B} - \frac{\alpha}{1-\alpha}\frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}B} + \frac{\mathrm{d}\log\kappa_{ss}^{\mathrm{canonical}}}{\mathrm{d}B},$$

where we can substitute $\frac{\mathrm{d}\log\ell(\kappa)}{\mathrm{d}B} = \frac{\ell'_\kappa\kappa}{\ell}\frac{\mathrm{d}\log\kappa}{\mathrm{d}B} + \frac{\ell'_B}{\ell}$ to obtain

$$\begin{aligned}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}B} &= -\frac{\alpha}{1-\alpha}\left[\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}B} + \frac{\ell'_B(\kappa_{ss})}{\ell(\bar{\kappa}_{ss})}\right] + \\ &-\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}B} + \frac{\ell'_B(\kappa_{ss})}{\ell(\bar{\kappa}_{ss})}\right] + \frac{\mathrm{d}\log\kappa_{ss}^{\mathrm{canonical}}}{\mathrm{d}B},\end{aligned} \tag{A.45}$$

where $\ell'_B(\bar{\kappa}_{ss})$ is the static derivative $\frac{\mathrm{d}\ell(\kappa;B)}{\mathrm{d}B}$ defined in Proposition 2, evaluated in the steady state $\bar{\kappa}_{ss}$. Recalling (40) and (40), the definitions $m_1 \equiv -\frac{\alpha}{1-\alpha}\frac{\ell'_\kappa\kappa}{\ell}$ and $m_2 \equiv -\frac{\ell'_\kappa\kappa}{\ell}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell}\frac{1}{1-\alpha}$ imply that (A.45) reduces to equation (48) in the text.

Derivation of (50). From definition (32), we have

$$\frac{\mathrm{d}}{\mathrm{d}n}\log\Gamma\left(\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\right) = -\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log[(1+n)\ell(\bar{\kappa}_{ss})]}{\mathrm{d}n},$$

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[39]The fact that $q_1 < 1-\alpha$ implies $\frac{\ell(\kappa)-q_1}{\ell(\kappa)-(1-\alpha)} > 1$. Also, the equilibrium restriction $\ell(\kappa) > (1-\alpha)$ implies $\frac{\ell^{\max}-(1-\alpha)}{\ell^{\max}-\ell(\kappa)} > 1$.

[40]Since condition (A.43) is equivalent to (A.36), satisfying (A.43) implies $m_2(\kappa) < 1-m_1(\kappa)$, where $m_1(\kappa) > 0$ under complementarity. Therefore, $m_2(\kappa) < 1$.

where both Γ and Γ' in the right hand side are evaluated in $\left(\bar{h}/\ell\left(\bar{\kappa}_{ss}\right)\right)$. Therefore, log-differentiating (45) with respect to n yields

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n} = -\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\frac{\mathrm{d}\log\left[(1+n)\ell\left(\bar{\kappa}_{ss}\right)\right]}{\mathrm{d}n} - \frac{\alpha}{1-\alpha}\frac{\mathrm{d}\log\ell\left(\bar{\kappa}_{ss}\right)}{\mathrm{d}n} + \frac{\mathrm{d}\log\kappa_{ss}^{\mathrm{canonical}}}{\mathrm{d}n}.$$

Substituting in the above expression the chain derivatives

$$\begin{aligned}\frac{\mathrm{d}\log[(1+n)\ell(\bar{\kappa}_{ss})]}{\mathrm{d}n} &= \frac{1}{1+n} + \frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}n},\\ \frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}n} &= \frac{\ell'_n(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})} + \frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n},\end{aligned}$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n} &= -\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{\ell'_n(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})}\right] - \frac{\alpha}{1-\alpha}\left[\frac{\ell'_n(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})}\right] + \\ &\quad -\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n}\right] - \frac{\alpha}{1-\alpha}\left[\frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n}\right] + \\ &\quad -\frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{1}{1+n}\right] + \frac{\mathrm{d}\log\kappa_{ss}^{\mathrm{canonical}}}{\mathrm{d}n}.\end{aligned} \quad (A.46)$$

Recalling (40) and (40), the definitions $m_1 \equiv -\frac{\alpha}{1-\alpha}\frac{\ell'_\kappa\kappa}{\ell}$ and $m_2 \equiv -\frac{\ell'_\kappa\kappa}{\ell}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell}\frac{1}{1-\alpha}$ imply that (A.46) reduces to

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n} = \frac{1}{1-(m_1+m_2)}\left[(m_1+m_2)\frac{\ell'_n}{\ell'_\kappa\kappa} + \frac{m_2}{1+n}\frac{\ell}{\ell'_\kappa\kappa} + \frac{\mathrm{d}\log\kappa_{ss}^{\mathrm{canonical}}}{\mathrm{d}n}\right],$$

where we can invert the sign of the variation $\mathrm{d}n$ to $-\mathrm{d}n$, and rearrange terms, to obtain equation (50) in the text.

Derivation of (51). From definition (32), we have

$$\frac{\mathrm{d}}{\mathrm{d}\bar{h}}\log\Gamma\left(\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\right) = \frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\frac{\mathrm{d}\log\left[\bar{h}/\ell\left(\bar{\kappa}_{ss}\right)\right]}{\mathrm{d}\bar{h}},$$

where both Γ and Γ' in the right hand side are evaluated in $\left(\bar{h}/\ell\left(\bar{\kappa}_{ss}\right)\right)$. Therefore, log-differentiating (45) with respect to $\bar{h}$ (recalling that $\mathrm{d}\kappa_{ss}^{\mathrm{canonical}}/\mathrm{d}\bar{h} = 0$) yields

$$\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}} = \frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell\left(\bar{\kappa}_{ss}\right)}\frac{\mathrm{d}\log\left[\bar{h}/\ell\left(\bar{\kappa}_{ss}\right)\right]}{\mathrm{d}\bar{h}} - \frac{\alpha}{1-\alpha}\frac{\mathrm{d}\log\ell\left(\bar{\kappa}_{ss}\right)}{\mathrm{d}\bar{h}}.$$

Substituting in the above expression the chain derivatives

$$\begin{aligned}\frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}\bar{h}} &= \frac{\ell'_{\bar{h}}(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})} + \frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}},\\ \frac{\mathrm{d}\log[\bar{h}/\ell(\bar{\kappa}_{ss})]}{\mathrm{d}\bar{h}} &= \frac{1}{\bar{h}} - \frac{\mathrm{d}\log\ell(\bar{\kappa}_{ss})}{\mathrm{d}\bar{h}},\end{aligned}$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}} &= \frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\frac{1}{\bar{h}} - \frac{1}{1-\alpha}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell(\bar{\kappa}_{ss})}\left[\frac{\ell'_{\bar{h}}(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})} + \frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}\bar{h}}\right] + \\ &\quad -\frac{\alpha}{1-\alpha}\left[\frac{\ell'_{\bar{h}}(\bar{\kappa}_{ss})}{\ell(\bar{\kappa}_{ss})} + \frac{\ell'_\kappa(\bar{\kappa}_{ss})\bar{\kappa}_{ss}}{\ell(\bar{\kappa}_{ss})}\frac{\mathrm{d}\log\bar{\kappa}_{ss}}{\mathrm{d}n}\right].\end{aligned} \quad (A.47)$$

Recalling (40) and (40), the definitions $m_1 \equiv -\frac{\alpha}{1-\alpha}\frac{\ell'_\kappa\kappa}{\ell}$ and $m_2 \equiv -\frac{\ell'_\kappa\kappa}{\ell}\frac{\Gamma'}{\Gamma}\frac{\bar{h}}{\ell}\frac{1}{1-\alpha}$ imply that (A.47) reduces to (51).

Introducing the welfare state: derivation of (53). We assume that the government taxes young agents' income at rate τ_t, and uses these revenues to pay for an amount g_t of care

(at the regular price p_t) under balanced budget, as established in condition (52). The private budget constraints thus read

$$c_t = w_t (1 - \tau_t) - s_t, \tag{A.48}$$

$$s_t R_{t+1} = d_{t+1} + p_{t+1} (h_{t+1} - g_{t+1}). \tag{A.49}$$

Since labor is supplied inelastically, the equilibrium in the labor market is not directly affected by the tax: from (12), (15) and (16), we obtain

$$p_t = (B/\eta) (1 - \alpha) (\kappa_t/\ell_t)^\alpha \equiv \Phi (\ell_t, \kappa_t). \tag{A.50}$$

which is the same relationship derived in the model without taxes – see expressions (21)-(22). From the modified household problem with constraints (A.48)-(A.49), utility maximization requires

$$c_t = \tfrac{1}{1+\beta} \left[w_t (1 - \tau_t) - \tfrac{p_{t+1}}{R_{t+1}} (\bar{h} - g_{t+1}) \right] \text{ and } s_t = \tfrac{1}{1+\beta} \left[\beta w_t (1 - \tau_t) + \tfrac{p_{t+1}}{R_{t+1}} (\bar{h} - g_{t+1}) \right] \tag{A.51}$$

and second-period consumption equals

$$d_t = (1 + n) [\ell_t - (1 - \alpha) (1 - \tau_t)] a_t^{1-\alpha} (\kappa_t/\ell_t)^\alpha, \tag{A.52}$$

Result (19) implies that second-period consumption is positive only if $\ell_t > (1 - \alpha) (1 - \tau_t)$, which will always turn out to be the case in equilibrium. Utility maximization also determines the relative demand for health care services:

$$\frac{p_t}{d_t} (h_t - \bar{h}) = \left(\frac{1 - \gamma}{\gamma} \right)^\sigma p_t^{1-\sigma}. \tag{A.53}$$

By inserting for d_t from (A.52) and p_t from (21) in the left hand side of (A.53), as well as $h_t = \eta (1 + n) (1 - \ell_t)$ and $\bar{h} = \eta (1 + n) (1 - \ell^{\max})$, we get

$$p_t = \left(\frac{1 - \gamma}{\gamma} \right)^{\frac{\sigma}{\sigma-1}} \left[\frac{(\ell^{\max} - \ell_t) (1 - \alpha)}{\ell_t - (1 - \alpha) (1 - \tau_t)} \right]^{\frac{1}{1-\sigma}} \equiv \Psi(\ell_t; \tau_t), \tag{A.54}$$

which is expression (53) in the text.

Introducing the welfare state: static equilibrium. Combining (A.54) and (21), the equilibrium employment share in the generic sector is the fixed point

$$\ell_t = \ell(\kappa_t; \tau_t) \equiv \arg \text{solve}_{\{\ell_t\}} [\Phi (\ell_t, \kappa_t) = \Psi(\ell_t; \tau_t)] \tag{A.55}$$

Note that (A.54) implies that $\Psi(\ell_t; \tau_t)$ is decreasing (increasing) in τ_t when $\sigma < 1$ ($\sigma > 1$). Therefore, recalling the graphical analysis of Figure 2, with the curve $\Psi(\ell_t)$ replaced by $\Psi(\ell_t; \tau_t)$, an increase in τ implies a leftward shift in the $\Psi(\ell_t; \tau_t)$ under both complementarity and substitutability. Consequently, the static effect of an increase in τ *for given* κ on sectoral labor shares is

$$\ell'_\tau \equiv \frac{d\ell(\kappa; \tau)}{d\tau} < 0. \tag{A.56}$$

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Introducing the welfare state: derivation of (54). Following the same steps as in equation (30), the workers' share of output

$$\frac{w_t N_t^y (1 - \tau_t)}{Y_t} = \frac{(1 - \tau_t)(1 - \alpha)\frac{x_t}{\ell_t} N_t^y}{x_t \left(\frac{1-\alpha}{\ell_t} + \alpha\right) N_t^y} = \frac{(1 - \alpha)(1 - \tau_t)}{1 - \alpha(1 - \ell_t)}, \tag{A.57}$$

Combining (A.57) with the saving function in (A.51), and substituting p_{t+1} and R_{t+1} by means of (15) and (16), we obtain

$$\frac{s_t}{w_t(1 - \tau_t)} = \frac{\beta}{1 + \beta}\left(1 + \frac{1}{\eta}\frac{1 - \alpha}{\alpha}\frac{\kappa_{t+1}}{\ell_{t+1}}\frac{(\bar{h} - g_{t+1})}{\beta w_t(1 - \tau_t)}\right). \tag{A.58}$$

Substituting $s_t = (1 + n)\kappa_{t+1}$ and inserting $\bar{h} = \eta(1 + n)(1 - \ell^{\max})$ from (13), as well as substituting g_{t+1} and w_t by means of (52) and (15), respectively, we get

$$\kappa_{t+1} = (1 - \tau_t)\frac{B\beta(1 - \alpha)}{(1 + \beta)(1 + n)}\kappa_t^\alpha \ell_t^{-\alpha}\left[1 - \frac{(1 - \alpha)(1 - \ell^{\max} - \tau_{t+1})}{\alpha(1 + \beta)\ell_{t+1}}\right]^{-1}. \tag{A.59}$$

Setting $\bar{h} = 0$ implies $\ell^{\max} = 1$, so that equation (A.59) reduces to expression (54) in the text.

Introducing the welfare state: long-run equilibrium

Under the assumption $(\bar{h} = 0, \ell^{\max} = 1)$, setting a constant tax rate $\tau_t = \tau$ in each t, and substituting the equilibrium condition (A.55) into (54), we have

$$\kappa_{t+1}\left[1 + \frac{(1 - \alpha)}{\alpha(1 + \beta)}\frac{\tau}{\ell(\kappa_{t+1}; \tau)}\right] = \tilde{B}\kappa_t^\alpha \ell(\kappa_t; \tau)^{-\alpha}(1 - \tau), \tag{A.60}$$

where we have defined $\tilde{B} \equiv \frac{B\beta(1-\alpha)}{(1+\beta)(1+n)}$. The dynamic stability of (A.60) requires that $\frac{d\kappa_{t+1}}{d\kappa_t}\frac{\kappa_t}{\kappa_{t+1}}$ evaluated in the steady state κ_{ss} is less than unity, i.e.[41]

$$-\frac{1}{1 - \alpha}\left(\alpha - \frac{\tilde{Q}(\kappa_{ss}; \tau)}{1 + \tilde{Q}(\kappa_{ss}; \tau)}\right)\frac{\ell'_{\kappa_{ss}}\kappa_{ss}}{\ell(\kappa_{ss}; \tau)} \equiv m_\tau(\kappa_{ss}) < 1, \tag{A.61}$$

where we have defined $\tilde{Q}(\kappa_{ss}; \tau) \equiv \frac{(1-\alpha)}{\alpha(1+\beta)}\frac{\tau}{\ell(\kappa_{ss};\tau)}$. Note that setting $\tau = 0$, the term $\tilde{Q}(\kappa_{ss}; \tau)$ reduces to zero and the term $m_\tau(\kappa_{ss})$ reduces to $m_1(\kappa_{ss})$ defined in (40). Therefore, condition (A.61) is the stability condition (40) generalized to the model with welfare state. The steady state of equation (A.60) is represented by

$$\kappa_{ss} = \tilde{B}^{\frac{1}{1-\alpha}}(1 - \tau)^{\frac{1}{1-\alpha}}\left[1 + \tilde{Q}(\kappa_{ss}; \tau)\right]^{-\frac{1}{1-\alpha}}\ell(\kappa_{ss}; \tau)^{-\frac{\alpha}{1-\alpha}}. \tag{A.62}$$

Log-differentiating (A.62) with respect to τ we have

$$\frac{d\log\kappa_{ss}}{d\tau} = -\frac{1}{1 - \alpha}\left[\frac{1}{1 - \tau} + \frac{d\log\left[1 + \tilde{Q}(\kappa_{ss}; \tau)\right]}{d\tau} + \alpha\frac{d\log\ell(\kappa_{ss}; \tau)}{d\tau}\right]. \tag{A.63}$$

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[41]Log-differentiating the left hand side of (A.60) yields $\frac{1}{\kappa_{t+1}} - \frac{\frac{(1-\alpha)}{\alpha(1+\beta)}\frac{\tau}{\ell(\kappa_{t+1};\tau)}}{1 + \frac{(1-\alpha)}{\alpha(1+\beta)}\frac{\tau}{\ell(\kappa_{t+1};\tau)}}\frac{\ell'_{\kappa_{t+1}}}{\ell(\kappa_{t+1};\tau)}$ whereas log-differentiating the right hand side gives $\alpha\frac{1}{\kappa_t} - \alpha\frac{\ell'_{\kappa_t}}{\ell(\kappa_t;\tau)}$. Taking the ratio and imposing the steady state $\kappa_{t+1} = \kappa_t = \kappa_{ss}$, the stability condition $\frac{d\kappa_{t+1}}{d\kappa_t}\frac{\kappa_t}{\kappa_{t+1}} < 1$ reduces to expression (A.61).

From the definition of $\tilde{Q}(\kappa_{ss}; \tau)$, we can substitute $\frac{\mathrm{d}\log[1+\tilde{Q}(\kappa_{ss};\tau)]}{\mathrm{d}\tau} = \frac{\tilde{Q}(\kappa_{ss};\tau)}{1+\tilde{Q}(\kappa_{ss};\tau)}\left(1 - \frac{\mathrm{d}\log\ell(\kappa_{ss};\tau)}{\mathrm{d}\tau}\right)$ in the above expression to obtain

$$\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}\tau} = -\frac{1}{1-\alpha}\left[\frac{1}{1-\tau} + \frac{\tilde{Q}(\kappa_{ss};\tau)}{1+\tilde{Q}(\kappa_{ss};\tau)} + \left(\alpha - \frac{\tilde{Q}(\kappa_{ss};\tau)}{1+\tilde{Q}(\kappa_{ss};\tau)}\right)\frac{\mathrm{d}\log\ell(\kappa_{ss};\tau)}{\mathrm{d}\tau}\right].$$

Further substituting the chain derivative $\frac{\mathrm{d}\log\ell(\kappa_{ss};\tau)}{\mathrm{d}\tau} = \frac{\ell'_{\kappa_{ss}}\kappa_{ss}}{\ell}\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}\tau} + \frac{\ell'_\tau}{\ell}$, we have

$$\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}\tau} = -\frac{1}{1-\alpha}\frac{1}{1-m_\tau(\kappa_{ss})}\left\{\left[\frac{1}{1-\tau} + \left(\alpha - \frac{\tilde{Q}(\kappa_{ss};\tau)}{1+\tilde{Q}(\kappa_{ss};\tau)}\right)\frac{\ell'_\tau(\kappa_{ss};\tau)}{\ell(\kappa_{ss};\tau)}\right] + \frac{\tilde{Q}(\kappa_{ss};\tau)}{1+\tilde{Q}(\kappa_{ss};\tau)}\right\}. \tag{A.64}$$

where $\ell'_\tau \equiv \frac{\mathrm{d}\ell(\kappa_{ss};\tau)}{\mathrm{d}\tau}$ is the derivative for given κ defined in (A.56) above, evaluated in κ_{ss}.

We next need to prove that a higher tax implies a lower steady state capital stock:

Proof. In order to prove that $\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}\tau} < 0$ we start by deriving $\ell'_\tau(\kappa_{ss};\tau)$. By implicit differentiation the definition of $\ell(\kappa_{ss};\tau)$ from (A.55) we get

$$\ell'_\tau(\kappa_{ss};\tau) = \frac{-\alpha\ell(1-\ell)(1-\alpha)}{(\tau+\alpha-\alpha\tau)\ell - \alpha(1-\ell)(\ell-(1-\tau)(1-\alpha))(1-\sigma)} < 0$$

where the sign follows from the fact that, as $\ell > (1-\alpha)(1-\tau)$, the denominator is increasing in σ, and that when $\sigma = 0$ it can be written

$$D \equiv \alpha\ell^2 + (1-\alpha)\ell\tau + (1-\alpha)\alpha(1-\tau)(1-\ell) > 0$$

If we can show that the curly bracket, $\{\dots\}$ in (A.64), cannot become negative we have proven that $\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}\tau} < 0$. First we note that

$$\alpha > \frac{\tilde{Q}(\kappa_{ss};\tau)}{1+\tilde{Q}(\kappa_{ss};\tau)}$$

is a necessary condition for the curly bracket in (A.64) to be negative. Then as $\tilde{Q}(\kappa_{ss};\tau)/\left(1+\tilde{Q}(\kappa_{ss};\tau)\right)$ is everywhere decreasing in β and as the absolute value of ℓ'_τ is everywhere decreasing in σ, the curly bracket in (A.64) cannot become negative for any β and σ unless it can become negative when β takes its largest allowed value of unity and σ takes its smallest allowed value of zero. We further note that a necessary condition for the curly bracket in (A.64) to be negative is that the bracket $[\dots]$ in (A.64) is negative. Inserting for $\beta = 1$ and $\sigma = 0$ we get

$$[\dots] = \frac{2\alpha^2\ell^3 + 3(1-\alpha)\alpha\ell 2 + \tau(1-\alpha)^2(1+2\tau\ell-\tau-\ell)}{D(1-\tau)(2\ell\alpha+(1-\alpha)\tau)} > 0$$

Both the numerator and the denominator are positive.[42] Hence we have proven that $[\dots]$ is positive for all τ, α and ℓ. Therefore $\{\dots\}$, the curly bracket in (A.64), is also always positive and thus $\frac{\mathrm{d}\log\kappa_{ss}}{\mathrm{d}\tau}$ is always negative. ∎

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[42]$(1+2\tau\ell-\tau-\ell)$ is a hyperbolic paraboloid and can be written as $(2(\tau-1/2)(\ell-1/2)+1/2)$ which cannot become negative for any ℓ and τ between zero and one.

Proof of Proposition 5. From (55), we have

$$\frac{\kappa_{t+1}}{\kappa_t} \gtreqless 1 \quad \Longleftrightarrow \quad \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)} \gtreqless \ell(\kappa_t), \tag{A.65}$$

which determines whether accumulation is positive, and

$$\frac{\mathrm{d}\frac{\kappa_{t+1}}{\kappa_t}}{\mathrm{d}\ell(\kappa_t)} < 0 \quad \text{and} \quad \frac{\mathrm{d}\frac{\kappa_{t+1}}{\kappa_t}}{\mathrm{d}\kappa_t} = \frac{\mathrm{d}\frac{\kappa_{t+1}}{\kappa_t}}{\mathrm{d}\ell(\kappa_t)}\ell'_{\kappa_t}, \tag{A.66}$$

which determines whether accumulation accelerates or decelerates. The sign of ℓ'_{κ_t} depends on the elasticity of substitution. From Proposition 1, we have $\ell'_\kappa < 0$ when $\sigma < 1$. Therefore, (A.66) implies that $\frac{\kappa_{t+1}}{\kappa_t}$ increases over time if $\kappa_{t+1}/\kappa_t > 1$, and decreases over time if $\kappa_{t+1}/\kappa_t < 1$. Depending on parameter values, we can distinguish among three cases.

(i) Suppose that there exists a finite critical level $\tilde{\kappa} > 0$ satisfying $\ell(\tilde{\kappa}) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}$ in an interior equilibrium. Existence requires that parameters satisfy $1-\alpha < \ell(\tilde{\kappa}) < 1$, and this condition implies

$$1-\alpha < \frac{(1+\beta)(1+n)}{A\beta} \quad \text{and} \quad \frac{(1+\beta)(1+n)}{A\beta} < 1. \tag{A.67}$$

In this case, it is possible that a finite positive initial endowment κ_0 is above or below $\tilde{\kappa}$. From (A.65), and the fact that $\ell'_\kappa < 0$, this yields three scenarios at time zero:

$$\begin{array}{ll}
\text{(i.a) If } \kappa_0 = \tilde{\kappa} \text{ then } \ell(\kappa_0) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)} \text{ and } \kappa_1 = \kappa_0 \\
\text{(i.b) If } \kappa_0 > \tilde{\kappa} \text{ then } \ell(\kappa_0) < \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)} \text{ and } \kappa_1 > \kappa_0 \\
\text{(i.c) If } \kappa_0 < \tilde{\kappa} \text{ then } \ell(\kappa_0) > \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)} \text{ and } \kappa_1 < \kappa_0
\end{array} \tag{A.68}$$

Since $\ell'_\kappa < 0$, scenario (i.a) implies constant $\kappa_t = \tilde{\kappa}$ forever; scenario (i.b) implies $\kappa_{t+1} > \kappa_t$ forever, with $\ell(\kappa_{t+1}) < \ell(\kappa_t)$ forever and the limiting results (56); scenario (i.c) implies $\kappa_{t+1} < \kappa_t$ forever, with $\ell(\kappa_{t+1}) > \ell(\kappa_t)$ forever and the limiting results (57).

(ii) Suppose that parameters satisfy $\frac{(1+\beta)(1+n)}{A\beta} < 1-\alpha < 1$. In this case, we have

$$1 < \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)} < \frac{A\beta}{(1+\beta)(1+n)} \tag{A.69}$$

and a hypothetical critical level $\tilde{\kappa}$ implying $\ell(\tilde{\kappa}) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}$ cannot be an interior equilibrium because the first inequality in (A.69) would imply $\ell(\tilde{\kappa}) > 1$, which is not feasible. This is equivalent to say that $\ell(\tilde{\kappa})$ is unfeasibly high – that is, $\tilde{\kappa}$ is unfeasibly low: given $\ell'_\kappa < 0$, any finite endowment $\kappa_0 > 0$ must be associated with an interior equilibrium

$$\ell(\kappa_0) < \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}$$

which implies self-reinforcing accumulation as in scenario (i.b) in expression (A.68).

(iii) Suppose that parameters satisfy $1 - \alpha < 1 < \frac{(1+\beta)(1+n)}{A\beta}$. In this case, we have

$$\frac{A\beta\,(1-\alpha)}{(1+\beta)(1+n)} < \frac{A\beta}{(1+\beta)(1+n)} < 1 \tag{A.70}$$

and a hypothetical critical level $\tilde{\kappa}$ implying $\ell(\tilde{\kappa}) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}$ cannot be an interior equilibrium because the last inequality in (A.70) would imply $\ell(\tilde{\kappa}) < 1 - \alpha$. This is equivalent to say that $\ell(\tilde{\kappa})$ is unfeasibly low – that is, $\tilde{\kappa}$ is unfeasibly high: given $\ell'_{\kappa} < 0$, any finite endowment $\kappa_0 > 0$ must be associated with an interior equilibrium

$$\ell(\kappa_0) > \frac{A\beta\,(1-\alpha)}{(1+\beta)(1+n)}$$

which implies self-reinforcing decumulation as in scenario (i.c) in expression (A.68). ∎

AK model: Proof of Proposition 6. From Proposition 1, we have $\ell'_{\kappa} > 0$ when $\sigma > 1$. Therefore, (A.66) implies that $\frac{\kappa_{t+1}}{\kappa_t}$ declines over time if $\kappa_{t+1}/\kappa_t > 1$, and increases over time if $\kappa_{t+1}/\kappa_t < 1$. Depending on parameter values, we can distinguish among three cases.

(I) Suppose that there exists a finite critical level $\hat{\kappa} > 0$ satisfying $\ell(\hat{\kappa}) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}$ in an interior equilibrium. Existence requires that parameters satisfy $1 - \alpha < \ell(\hat{\kappa}) < 1$, and this condition implies

$$1 - \alpha < \frac{(1+\beta)(1+n)}{A\beta} \quad \text{and} \quad \frac{(1+\beta)(1+n)}{A\beta} < 1. \tag{A.71}$$

In this case, it is possible that a finite positive initial endowment κ_0 is above or below $\hat{\kappa}$. From (A.65), and the fact that $\ell'_{\kappa} > 0$, this yields three scenarios at time zero:

$$\begin{array}{ll}
\text{(I.a) If } \kappa_0 = \hat{\kappa} \text{ then } \ell(\kappa_0) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)} \text{ and } \kappa_1 = \kappa_0 \\
\text{(I.b) If } \kappa_0 < \hat{\kappa} \text{ then } \ell(\kappa_0) < \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)} \text{ and } \kappa_1 > \kappa_0 \\
\text{(I.c) If } \kappa_0 > \hat{\kappa} \text{ then } \ell(\kappa_0) > \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)} \text{ and } \kappa_1 < \kappa_0
\end{array} \tag{A.72}$$

Since $\ell'_{\kappa} > 0$, scenario (I.a) implies constant $\kappa_t = \hat{\kappa}$ forever; scenario (I.b) implies $\kappa_{t+1} > \kappa_t$ and $\ell(\kappa_{t+1}) > \ell(\kappa_t)$ during the transition, and the asymptotic steady state $\lim_{t\to\infty} \kappa_t = \hat{\kappa}$ and $\lim_{t\to\infty} \ell(\kappa_t) = \ell(\hat{\kappa})$; scenario (I.c) implies $\kappa_{t+1} < \kappa_t$ and $\ell(\kappa_{t+1}) < \ell(\kappa_t)$ during the transition, and the asymptotic steady state $\lim_{t\to\infty} \kappa_t = \hat{\kappa}$ and $\lim_{t\to\infty} \ell(\kappa_t) = \ell(\hat{\kappa})$.

(II) Suppose that parameters satisfy $\frac{(1+\beta)(1+n)}{A\beta} < 1 - \alpha < 1$. In this case, we have

$$1 < \frac{A\beta\,(1-\alpha)}{(1+\beta)(1+n)} < \frac{A\beta}{(1+\beta)(1+n)} \tag{A.73}$$

and a hypothetical critical level $\hat{\kappa}$ implying $\ell(\hat{\kappa}) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}$ cannot be an interior equilibrium because the first inequality in (A.73) would imply $\ell(\hat{\kappa}) > 1$, which is not feasible. This is equivalent to say that $\ell(\hat{\kappa})$ is unfeasibly high – that is, $\hat{\kappa}$ is unfeasibly high: given $\ell'_{\kappa} > 0$, any finite endowment $\kappa_0 > 0$ must be associated with an interior equilibrium

$$\ell(\kappa_0) < \frac{A\beta\,(1-\alpha)}{(1+\beta)(1+n)}$$

which implies permanent (but decelerating) positive accumulation as in scenario (I.b) in expression (A.72).

(III) Suppose that parameters satisfy $1 - \alpha < 1 < \frac{(1+\beta)(1+n)}{A\beta}$. In this case, we have

$$\frac{A\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)} < \frac{A\beta}{\left(1+\beta\right)\left(1+n\right)} < 1 \qquad (A.74)$$

and a hypothetical critical level $\hat{\kappa}$ implying $\ell\left(\hat{\kappa}\right) = \frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}$ cannot be an interior equilibrium because the last inequality in (A.74) would imply $\ell\left(\hat{\kappa}\right) < 1 - \alpha$. This is equivalent to say that $\ell\left(\hat{\kappa}\right)$ is unfeasibly low – that is, $\tilde{\kappa}$ is unfeasibly low: given $\ell'_{\kappa} > 0$, any finite endowment $\kappa_0 > 0$ must be associated with an interior equilibrium

$$\ell\left(\kappa_0\right) > \frac{A\beta\left(1-\alpha\right)}{\left(1+\beta\right)\left(1+n\right)}$$

which implies permanent (but decelerating) decumulation as in scenario (I.c) in expression (A.72).

B   Further Details

Proposition 4: Further Details on Existence and Uniqueness of the steady state. To prove existence, we transform equation (35) into an equivalent dynamic law that maps $\ell\left(\kappa_t\right)$ into $\ell\left(\kappa_{t+1}\right)$. Starting from expression (35), substitute $p_t = \Phi\left(\ell_t, \kappa_t\right) = \left(B/\eta\right)\left(1-\alpha\right)\left(\kappa_t/\ell_t\right)^\alpha$ from (22) to write

$$\frac{\kappa_{t+1}}{\ell\left(\kappa_{t+1}\right)}\left[\ell\left(\kappa_{t+1}\right) - \frac{\left(1-\alpha\right)\left(1-\ell^{\max}\right)}{\alpha\left(1+\beta\right)}\right] = \frac{\eta\beta}{\left(1+\beta\right)\left(1+n\right)}p_t. \qquad (B.1)$$

Imposing the equilibrium condition $p_t = p\left(\kappa_t\right) \equiv \Psi\left(\ell\left(\kappa_t\right)\right)$ from (26), we have

$$\frac{\kappa_{t+1}}{\ell\left(\kappa_{t+1}\right)}\left[\ell\left(\kappa_{t+1}\right) - \frac{\left(1-\alpha\right)\left(1-\ell^{\max}\right)}{\alpha\left(1+\beta\right)}\right] = \frac{\eta\beta}{\left(1+\beta\right)\left(1+n\right)}\Psi\left(\ell\left(\kappa_t\right)\right). \qquad (B.2)$$

Also notice that, setting (22) at time $t+1$ and solving for the input ratio, we have $\frac{\kappa_{t+1}}{\ell_{t+1}} = \left[\frac{p_{t+1}}{(B/\eta)(1-\alpha)}\right]^{\frac{1}{\alpha}}$. Plugging this result in (B.2), and imposing the static equilibrium condition $p_{t+1} = p\left(\kappa_{t+1}\right) \equiv \Psi\left(\ell\left(\kappa_{t+1}\right)\right)$ from (26), we obtain the *dynamic law*

$$\Psi\left(\ell\left(\kappa_{t+1}\right)\right)\left(\ell\left(\kappa_{t+1}\right) - q_1\right)^\alpha = q_2\Psi\left(\ell\left(\kappa_t\right)\right)^\alpha, \qquad (B.3)$$

where we again have used that the definition of q_1 and also have defined the convenient variable q_2:

$$q_1 \equiv \frac{(1-\alpha)(1-\ell^{\max})}{\alpha(1+\beta)} > 0 \quad \text{and} \quad q_2 \equiv \frac{B}{\eta}\left(1-\alpha\right)\left[\frac{\eta\beta}{(1+\beta)(1+n)}\right]^\alpha > 0. \qquad (B.4)$$

Expression (B.3) fully characterizes the dynamics of capital per worker. Dynamics are well defined only if both sides are strictly positive, which requires $\ell\left(\kappa_{t+1}\right) > q_1$ in each period $t+1$:

this inequality is always satisfied as shown in (A.42). In (B.3), the steady state condition $\kappa_{t+1} = \kappa_t = \bar{\kappa}_{ss}$ is satisfied when

$$\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \underbrace{\left[q_2\left(\ell\left(\bar{\kappa}_{ss}\right) - q_1\right)^{-\alpha}\right]^{\frac{1}{1-\alpha}}}_{\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}. \tag{B.5}$$

For future reference, we define the elasticities of $\Psi\left(.\right)$ and $\Omega\left(.\right)$ with respect to $\bar{\kappa}_{ss}$ as

$$\epsilon_1 \equiv \frac{\mathrm{d}\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss}}\frac{\bar{\kappa}_{ss}}{\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)} = \frac{\Psi'\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}{\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}\ell'_\kappa\left(\bar{\kappa}_{ss}\right)\bar{\kappa}_{ss}, \tag{B.6}$$

$$\epsilon_2 \equiv \frac{\mathrm{d}\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss}}\frac{\bar{\kappa}_{ss}}{\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right)} = -\frac{\frac{\alpha}{1-\alpha}}{\ell\left(\bar{\kappa}_{ss}\right) - q_1}\ell'_\kappa\left(\bar{\kappa}_{ss}\right)\bar{\kappa}_{ss}. \tag{B.7}$$

The remainder of the proof studies separately the two cases of substitutability and complementarity.

Substitutability. When $\sigma > 1$, results (A.22) imply that

$$\frac{\mathrm{d}\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss}} = \underbrace{\Psi'\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}_{\text{positive}}\underbrace{\ell'_\kappa\left(\bar{\kappa}_{ss}\right)}_{\text{positive}} > 0 \text{ and } \ell'\left(\bar{\kappa}_{ss}\right) > 0. \tag{B.8}$$

Result (B.8) implies that, given the definitions in (B.5), function $\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ is strictly increasing in $\bar{\kappa}_{ss}$ whereas $\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ is strictly decreasing in $\bar{\kappa}_{ss}$. Using the limiting properties (A.21) and (A.24), we also have

$$\begin{aligned} &\lim_{\bar{\kappa}_{ss}\to 0}\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = 0, &&\lim_{\bar{\kappa}_{ss}\to 0}\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \left[q_2/\left(1 - \alpha - q_1\right)^\alpha\right]^{\frac{1}{1-\alpha}} > 0, \\ &\lim_{\bar{\kappa}_{ss}\to\infty}\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \infty, &&\lim_{\bar{\kappa}_{ss}\to\infty}\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \left[q_2/\left(\ell^{\max} - q_1\right)^\alpha\right]^{\frac{1}{1-\alpha}} > 0. \end{aligned} \tag{B.9}$$

These results imply that, under substitutability, there exists a unique steady state $\bar{\kappa}_{ss}$ satisfying condition (B.5).

Complementarity. When $\sigma < 1$, results (A.22) imply that

$$\frac{\mathrm{d}\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss}} = \underbrace{\Psi'\left(\ell\left(\bar{\kappa}_{ss}\right)\right)}_{\text{negative}}\underbrace{\ell'_\kappa\left(\bar{\kappa}_{ss}\right)}_{\text{negative}} > 0 \text{ and } \ell'_\kappa\left(\bar{\kappa}_{ss}\right) < 0. \tag{B.10}$$

Result (B.10) implies that, given the definitions in (B.5), both $\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ and $\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ in (B.5) are strictly increasing in $\bar{\kappa}_{ss}$. Using the limiting properties (A.21) and (A.24), we also have

$$\begin{aligned} &\lim_{\bar{\kappa}_{ss}\to 0}\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = 0, &&\lim_{\bar{\kappa}_{ss}\to 0}\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \left[q_2/\left(\ell^{\max} - q_1\right)^\alpha\right]^{\frac{1}{1-\alpha}} > 0, \\ &\lim_{\bar{\kappa}_{ss}\to\infty}\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \infty, &&\lim_{\bar{\kappa}_{ss}\to\infty}\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \left[q_2/\left(1 - \alpha - q_1\right)^\alpha\right]^{\frac{1}{1-\alpha}} > 0. \end{aligned} \tag{B.11}$$

The limits in (B.11) imply that *there always exists at least one steady state $\bar{\kappa}_{ss(1)}$ satisfying condition (B.5) in which $\Psi\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ cuts $\Omega\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$ from below:* this steady state therefore satisfies

$$\frac{\mathrm{d}\Psi\left(\ell\left(\bar{\kappa}_{ss(1)}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss(1)}} > \frac{\mathrm{d}\Omega\left(\ell\left(\bar{\kappa}_{ss(1)}\right)\right)}{\mathrm{d}\bar{\kappa}_{ss(1)}}. \tag{B.12}$$

Considering stability, equation (B.3) implies that any steady state $\bar{\kappa}_{ss(i)}$ is stable if

$$\frac{\Psi'\left(\ell\left(\bar{\kappa}_{ss(i)}\right)\right)}{\Psi\left(\ell\left(\bar{\kappa}_{ss(i)}\right)\right)} < -\frac{\alpha}{1-\alpha}\frac{1}{\ell\left(\bar{\kappa}_{ss(i)}\right) - q_1}. \tag{B.13}$$

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It is easily shown that (B.12) implies that the steady state $\bar{\kappa}_{ss(1)}$ satisfies the stability condition (B.13). Hence, under complementarity, there always exist a stable steady state $\bar{\kappa}_{ss(1)}$. In order to assess the uniqueness of the steady state, re-write the steady-state condition (B.5) in explicit form by substituting $\Psi\left(\cdot\right)$ with (24), obtaining

$$\ell\left(\bar{\kappa}_{ss}\right) - q_1 = \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\sigma(1-\alpha)}{\alpha(1-\sigma)}} q_2^{\frac{1}{\alpha}} \left[\frac{\ell\left(\bar{\kappa}_{ss}\right) - (1-\alpha)}{(1-\alpha)\left(\ell^{\max} - \ell\left(\bar{\kappa}_{ss}\right)\right)}\right]^{\frac{1-\alpha}{\alpha(1-\sigma)}}. \tag{B.14}$$

Hence, defining $q_3 \equiv \left(\frac{1-\gamma}{\gamma}\right)^{\frac{\sigma(1-\alpha)}{\alpha(1-\sigma)}} q_2^{\frac{1}{\alpha}}$, the steady-state condition reads

$$\ell\left(\bar{\kappa}_{ss}\right) = F\left(\ell\left(\bar{\kappa}_{ss}\right)\right) \quad \text{where} \quad F\left(\ell\left(\bar{\kappa}_{ss}\right)\right) \equiv q_1 + q_3 \left[\frac{\ell\left(\bar{\kappa}_{ss}\right) - (1-\alpha)}{(1-\alpha)\left(\ell^{\max} - \ell\left(\bar{\kappa}_{ss}\right)\right)}\right]^{\frac{1-\alpha}{\alpha(1-\sigma)}}. \tag{B.15}$$

In general, the function $F\left(\ell\right)$ is strictly increasing and exhibits the following properties

$$\lim_{\ell \to 1-\alpha} F\left(\ell\right) = q_1 < \ell \text{ and } \lim_{\ell \to \ell^{\max}} F\left(\ell\right) = \infty, \tag{B.16}$$

$$F'\left(\ell\right) = \frac{q_3}{\alpha\left(1-\sigma\right)} \left[\frac{\ell - (1-\alpha)}{(1-\alpha)\left(\ell^{\max} - \ell\right)}\right]^{\frac{1-\alpha}{\alpha(1-\sigma)}-1} \frac{\ell^{\max} - (1-\alpha)}{\left(\ell^{\max} - \ell\right)^2} > 0. \tag{B.17}$$

Evaluating $F'\left(\ell\right)$ in a steady state $\ell\left(\bar{\kappa}_{ss}\right) = F\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$, we have

$$F'\left(\ell\left(\bar{\kappa}_{ss}\right)\right) = \frac{1-\alpha}{\alpha\left(1-\sigma\right)} \underbrace{\frac{\ell\left(\bar{\kappa}_{ss}\right) - q_1}{\ell\left(\bar{\kappa}_{ss}\right) - (1-\alpha)}}_{>1} \underbrace{\frac{\ell^{\max} - (1-\alpha)}{\ell^{\max} - \ell\left(\bar{\kappa}_{ss}\right)}}_{>1}. \tag{B.18}$$

Comparing (B.18) with (A.43), it is evident that the stability condition (A.43) is equivalent to $F'\left(\ell\left(\bar{\kappa}_{ss}\right)\right) > 1$. In graphical terms, this means that a stable steady state is a an intersection $\ell = F\left(\ell\right)$ in which the function $F\left(\ell\right)$ cuts the 45-degree line $\ell = \ell$ from below – e.g., like the steady state shown in Figure 4, graph (a). Properties (B.16)-(B.17) thus confirm the existence of at least one stable steady state. Concerning the uniqueness of the steady state, we must consider two sub-cases, depending on whether the parameter values imply $\frac{1-\alpha}{\alpha(1-\sigma)} > 1$ or $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$.

Subcase I. When $\frac{1-\alpha}{\alpha(1-\sigma)} > 1$, expression (B.17) implies $F''\left(\ell\right) > 0$ for all $\ell \in (1-\alpha, \ell^{\max})$, so that $F\left(\ell\right)$ is strictly increasing and strictly convex for all $\ell \in (1-\alpha, \ell^{\max})$. This means that there is a unique steady state $\ell\left(\bar{\kappa}_{ss}\right) = F\left(\ell\left(\bar{\kappa}_{ss}\right)\right)$, as shown in Figure 4, graph (a). Moreover, $\ell\left(\bar{\kappa}_{ss}\right)$ is stable, as is immediately evident from (B.18): when $\frac{1-\alpha}{\alpha(1-\sigma)} > 1$, all the three terms at the right hand side are strictly greater than unity, implying $F'\left(\ell\left(\bar{\kappa}_{ss}\right)\right) > 1$. Recalling that $\ell'_\kappa\left(\kappa\right) < 0$ under complementarity, an initial condition $\kappa\left(0\right) < \bar{\kappa}_{ss}$ implies positive accumulation and declining employment in generic production: the economy starts from an initial level $\ell_0 = \ell\left(\kappa\left(0\right)\right)$ and then declines towards $\ell_{ss} = \ell\left(\bar{\kappa}_{ss}\right)$ as shown in Figure 4, graph (a).

Subcase II. When $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$, we have $\lim_{\ell \to 1-\alpha} F'\left(\ell\right) = \infty$ and $\lim_{\ell \to \ell^{\max}} F'\left(\ell\right) = \infty$. Expression (B.17) implies that $F\left(\ell\right)$ is initially concave and then convex: from

$$\frac{F''\left(\ell\right)}{F'\left(\ell\right)} = \frac{1}{\ell^{\max} - \ell} \left\{2 - \left(1 - \frac{1-\alpha}{\alpha\left(1-\sigma\right)}\right) \frac{\ell^{\max} - (1-\alpha)}{\ell - (1-\alpha)}\right\}, \tag{B.19}$$

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there exists a point of inflection

$$\tilde{\ell} \equiv (1 - \alpha) + \frac{1}{2}\left(1 - \frac{1 - \alpha}{\alpha(1 - \sigma)}\right)\left[\ell^{\max} - (1 - \alpha)\right]$$

such that $F''\left(\hat{\ell}\right) = 0$, with $F''(\ell)$ is negative for $\ell < \tilde{\ell}$ and positive $\tilde{\ell} > \ell$. This implies that, in the subcase $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$, we may have in principle two possible outcomes: a unique stable steady state or three steady states, as shown in Figure 4, graphs (b) and (c). When there are three steady states, $\bar{\kappa}_{ss(1)} < \bar{\kappa}_{ss(2)} < \bar{\kappa}_{ss(3)}$, the middle steady state $\bar{\kappa}_{ss(2)}$ is unstable because $F'\left(\ell\left(\bar{\kappa}_{ss(2)}\right)\right) < 1$, whereas $\bar{\kappa}_{ss(1)}$ and $\bar{\kappa}_{ss(3)}$ are both stable. This scenario is thus characterized by

$$F\left(\ell_{ss3}\right) \quad < \quad F\left(\ell_{ss2}\right) < F\left(\ell_{ss1}\right), \tag{B.20}$$

$$F'\left(\ell_{ss3}\right) \quad < \quad 1, \quad F'\left(\ell_{ss2}\right) > 1, \quad F\left(\ell_{ss1}\right) < 1, \tag{B.21}$$

where $\ell_{ss}^i \equiv \ell\left(\bar{\kappa}_{ss(i)}\right)$. Recalling that $\ell_{\kappa}'(\kappa) < 0$ under complementarity, an initial condition $\kappa(0) < \bar{\kappa}_{ss(1)}$ implies positive accumulation and declining employment in generic production: the economy starts from an initial level $\ell_0 = \ell(\kappa(0))$ and then declines towards $\ell_{ss}^1 = \ell\left(\bar{\kappa}_{ss(1)}\right)$ as shown in Figure 4, graph (d).
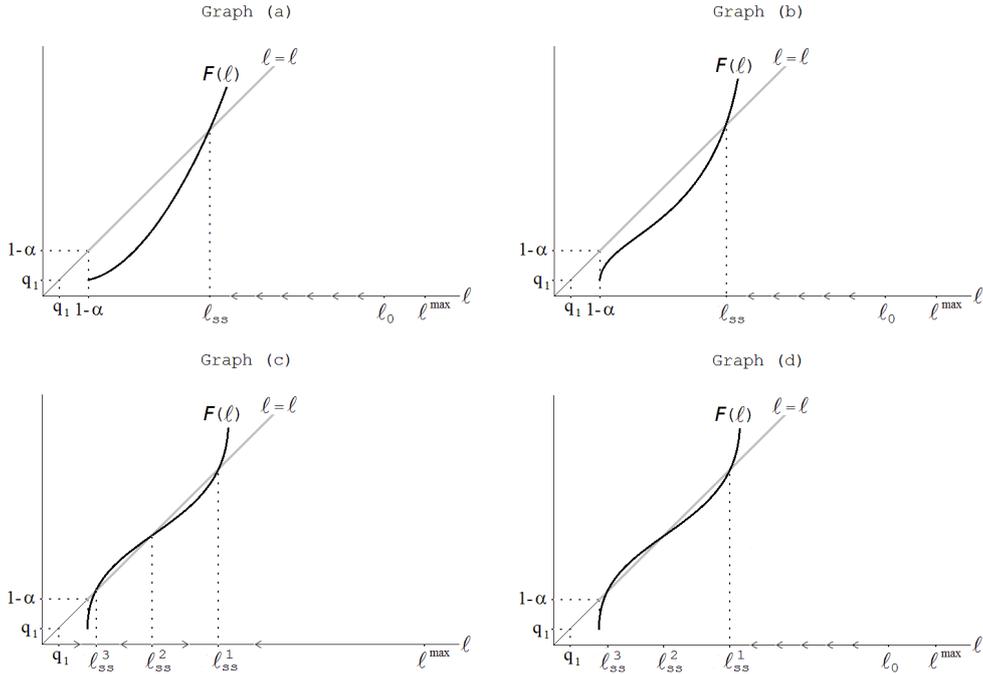

Figure 4: Existence and uniqueness of steady states under complementarity. Graph (a): the subcase $\frac{1-\alpha}{\alpha(1-\sigma)} > 1$ features a unique stable steady state. Graph (b): the subcase $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$ when the steady state is unique. Graphs (c)-(d): the subcase $\frac{1-\alpha}{\alpha(1-\sigma)} < 1$ when the steady states are three – the middle one being unstable.